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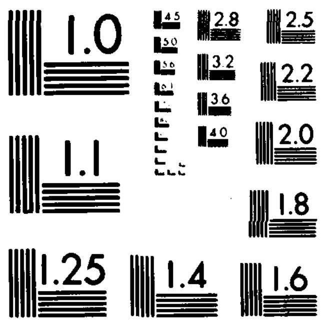


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(9) MRC Technical Summary Report, # 2024

(6) EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINEAR INTERPOLATING SPLINE CURVES.

(14) MRC-TSR-2024


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(11) Nov[redacted]79

(12) 70

(Received September 4, 1979)

(15) DAAG29-75-C-0024,
NSF-MPS74-02292




Approved for public release
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Sponsored by

U. S. Army Research Office
P. O. Box 12211
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UNIVERSITY OF WISCONSIN - MADISON
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# EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINEAR INTERPOLATING SPLINE CURVES

Michael Golomb[1] and Joseph Jerome[2]

Technical Summary Report # 2024

November 1979

## ABSTRACT

A detailed global and local analysis of smooth solutions of the variational problem

(1i) $$\delta \int_0^{\bar{s}} \kappa^2(s)ds = 0 \quad ,$$

subject to position function constraints

(1ii) $$x(s_i) = p_i, \quad 0 \le s_0 < s_1 < \cdots < s_m \le \bar{s} \quad ,$$

is carried out. Here $\{p_i\}_0^m \subset \mathbb{R}^2$ is prescribed, $x$ is a vector-valued function with curvature $\kappa(s)$ at arc length $s$ and the interpolation nodes $s_i$ are free. Problem (1) may be viewed as the mathematical formulation of the draftsman's technique of curve fitting by mechanical splines.

Although most of the basic equations satisfied by these nonlinear spline curves have been known for a very long time, calculation via elliptic integral functions has been hampered by a lack of understanding concerning what precise information must be specified for the stable determination of a smooth, unique interpolant modelling the thin elastic beam. In this report, sharp characterizations are derived for the extremal interpolants as well as structure theorems in terms of inflection point modes which guarantee uniqueness and well-posedness.

A certain type of stability is introduced and studied and shown to be related to (linearization) concepts associated with piecewise cubic spline

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Sponsored by the United States Army under Contract No. DAAG29-75-C-0024. This material is based upon work supported by the National Science Foundation under Grant No. MPS 74-02292 A01.

ABSTRACT (continued)

functions, which have been studied for decades as a simplification of the nonlinear spline curves. Many examples are introduced and studied.

AMS (MOS) Subject Classifications: 41A05, 49B15, 49B50, 49F22, 65D10

Key Words: Nonlinear spline curves, elastica, manifolds of extremals, perturbation stability, mode, ray configuration, rectangular configuration.

Work Unit Number 6 - Spline Functions and Approximation Theory

## SIGNIFICANCE AND EXPLANATION

The mathematical formulation of curve fitting by mechanical splines, i.e. thin, flexible, elastic beams passing through freely rotating sleeves anchored at fixed locations, is studied in this report. These are called elastica or nonlinear spline curves.

As contrasted with the mathematically idealized splines, which have proven to be of considerable utility and concerning which much information is available, the nonlinear splines are relatively poorly understood. The writers are attempting to understand and systematically construct these curves. Computer graphics obtained by other workers suggest remarkable efficiency of the elastica for curve fitting. This is perhaps not too surprising since the nonlinear spline represents an equilibrium position of a thin beam.



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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

# EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINEAR INTERPOLATING SPLINE CURVES

Michael Golomb[1] and Joseph Jerome[2]

## §1. Introduction

Let $P = \{p_0, p_1, \ldots, p_m\}$ be an ordered set of points in the Euclidean plane (the $p_i$ need not be distinct) and let it be required to pass a smooth curve through these points in the prescribed order. It is an old technique of draftsmen to use a mechanical spline to accomplish this. If the spline is considered as a thin elastic beam of uniform cross section with a central fiber that is inextensible, then the strain energy of the bent spline of length $\bar{s}$ is given by

$$A \int_0^{\bar{s}} \kappa^2(s)ds + B$$

where $\kappa(s)$ is the curvature of the fiber at arc length $s$ and $A, B$ are constants. An equilibrium position of the spline makes the energy functional stationary, hence satisfies

$$\delta \int_0^{\bar{s}} \kappa^2(s)ds = 0 \quad . \tag{1.1i}$$

This equation together with the interpolation conditions

$$x(s_i) = p_i, \quad 0 \le s_0 < s_1 < \cdots < s_m \le \bar{s} \tag{1.1ii}$$

for the position function $x$, constitute the mathematical formulation of the draftsman's technique. The present article deals with analytical (not graphical nor computational) problems arising from system (1.1). In elasticity theory the solutions of (1.1i) or of the more general equation

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Sponsored by the United States Army under Contract No. DAAG29-75-C-0024. This material is based upon work supported by the National Science Foundation under Grant No. MPS 74-02292 A01.

$$(1.2) \qquad \delta \left( \int_0^{\bar{s}} [\lambda + \kappa^2(s)]ds \right) = 0 \quad ,$$

where $\lambda$ is a constant, are known as elastica. Their study dates back to the Bernoulli brothers, Euler and others (see Love [8, Ch. XIX] for classical results). The boundary conditions in traditional elasticity theory have little in common with the interpolation conditions (1.1ii). To materialize the latter ones in the beam model one may think of freely rotating small sleeves, anchored at the points $p_0,\ldots,p_m$, through which the spline can slide without friction. We refer to the solutions $x = \tilde{x}$ of the variational problem (1.1i, ii) as extremal P-interpolants. In some parts of the present paper we deal with extremal length-prescribed P-interpolants, in which $\bar{s} = s_m - s_0$ is given in addition to P. To materialize this condition one replaces the sleeves at $p_0, p_m$ by pins which allow no sliding. In other parts we consider extremal angle-prescribed P-interpolants, in which the angles that the spline makes at $p_0$ and $p_m$ with a reference line are given. This situation prevails if the sleeves at $p_0$ and $p_m$ are not allowed to rotate.

Stable equilibrium positions in mechanics are sought as positions that minimize the potential energy functional. In one of the earliest papers discussing nonlinear interpolating splines [2] it was pointed out that the infimum of $\int_0^{\bar{s}} \kappa^2(s)ds$ is 0 for any configuration P, hence can be attained only in the trivial case where P is interpolated by a straight segment. Lee and Forsythe [7], who make a substantial study of the variational problem (1.1i, ii), call the solutions (when existence is hypothesized) local minima. However, it will be proved in §6 below that, in the simple case where P consists of 2 points, there are countably many nontrivial extremal P-interpolants; none of them constitutes a local minimum of the energy. This makes it evident that an extremal P-interpolant is, in general, not a local minimum (for a detailed discussion of the stability problem see [5]).

The existence questions for interpolating elastica are much more subtle. For length-constrained or length-prescribed interpolants one can prove existence of extremals (actually global minima) by the direct methods of the calculus of variations, because one has compactness in a suitably chosen function space (this was done in [3] and [6]; see also the

Appendix of this paper). This is not the case for interpolants with no length restriction, and the existence of such extremals interpolating $n$ points in general position, and whether they are local minima or not remains an open question (some progress along these lines has been achieved by M. Golomb [4], [5]). Computational work on extremal interpolants is more advanced (cf. M. Malcolm [10]), although decisive progress in this area is also hampered by the lack of general existence and uniqueness theorems.

We now give a brief account of the content of this paper. In §2 we define the function classes in which the extremal interpolants are sought. We also characterize them by Euler equations (for the Cartesian coordinates), boundary and regularity conditions. In §3 we do the same for the "normal representation" of the extremals, by which we mean the function $s \mapsto \theta(s)$, which is the angle that the extremal makes at arc length $s$ with a reference line.

The normal representation $\theta$ of a length-prescribed extremal P-interpolant appears as the solution of a free multi-point boundary value problem for $s_1,\ldots,s_{m-1}$, $\theta$ with $s_m - s_0$ prescribed:

$$\text{(1.3)}\quad \begin{aligned} &\text{(i)} && \ddot{\theta}(s) - \mu_i^1 \sin\theta(s) + \mu_i^2 \cos\theta(s) = 0, \quad s_{i-1} < s < s_i \;, \\ &\text{(ii)} && \dot{\theta}(s_0) = \dot{\theta}(s_m) = 0 \;, \\ &\text{(iii)} && (\mu_{i+1}^1 - \mu_i^1)\cos\theta(s_i) + (\mu_{i+1}^2 - \mu_i^2)\sin\theta(s_i) = 0, \quad i = 1,\cdots,m-1 \;. \end{aligned}$$

For the general extremal P-interpolant, (1.3iii) holds also for $i = m$ and $s_0$ and $s_m$ are free as well with $\mu_{m+1}^1 = \mu_{m+1}^2 = 0$. The function $\theta$ and the knot abscissas $s_i$ are the unknowns; the multipliers $\mu_i^1$, $\mu_i^2$ are determined from the interpolation conditions. In §4 it is shown how certain families $\mathcal{E}_{(k_1,\ldots,k_m)}$ of extremal interpolants with prescribed numbers $(k_1,\ldots,k_m)$ of inflection points between knots ("mode"), can be realized as smooth 2m-dimensional manifolds $\mathcal{M}_{(k_1,\ldots k_m)}$. The inverse mapping from $\mathcal{M}_{(k_1,\ldots k_m)}$ into the position function space of the extremals is continuous when the latter is topologized by a suitable metric. Thus, the mode $(k_1,\ldots,k_m)$ of an extremal suitably delineates uniqueness and well-posedness. The union of the $\mathcal{E}_{(k_1,\ldots,k_m)}$ consists of only those extremals which have a genuine knot and no inflection point at each interior interpolation node. Points in the intersection of the boundaries of the manifolds $M_{(k_1,\ldots,k_m)}$ correspond to singular points in position function space. For these boundary elements some interior knot is spurious ($\kappa'$ is not discontinuous) or is an inflection point. We give two examples to demonstrate this.

In §5 we study the existence of elastica spline interpolation in the small. Does the set of configurations P for which extremal interpolants exist have nonempty interior in $\mathbb{R}^{2m}$? More specifically, which P in $\mathbb{R}^{2m}$ are interior points of this set? We show, by use of the implicit function theorem that near a given configuration $\bar{P}$ with extremal interpolant $\bar{E}$ there is a local diffeomorphism between configurations and extremal interpolants if $(\bar{E},\bar{P})$ satisfies a certain hypothesis (A). It requires that a homogeneous linear differential equation with variable coefficients depending on $\bar{E}$ and with homogeneous linear side conditions has no nontrivial solution. Another formulation of this condition is that a computable function (involving many quadratures) be $\neq 0$ at the end point of $\bar{E}$. It is easily verified that the ray configuration $P_0$, with the trivial extremal interpolant $E_0$, satisfies (A), so that the existence of extremal P-interpolants for all configurations P in some Euclidean neighborhood of any ray configuration is thereby demonstrated. The differential equation problem of hypothesis (A) reduces to the natural cubic spline interpolation problem in the case $(P_0,E_0)$. This demonstrates that cubic spline interpolation can be interpreted as the result of linearization of extremal interpolation (in the sense of making $\int\kappa^2 ds$ stationary) near the trivial interpolant for the ray configuration. This proof makes precise the old idea that cubic splines are in some sense the "smoothest" interpolants. Of course, it has long been known that cubic spline functions arise from minimizing the quadratic functional $\int(D^2f)^2$ among the interpolating functions f. Since the linear operator $D^2$ supposedly approximates the nonlinear curvature operator, the cubic splines recommend themselves as near optimally smooth interpolants. The "hairpin" confirguration $\tilde{P}$ with a loop interpolant $\tilde{E}$ is given as an example where hypothesis (A) is not satisfied. There are configurations close to $\tilde{P}$ for which there exists no extremal interpolant near $\tilde{E}$ and there are other configurations close to $\tilde{P}$ for which there do exist extremal interpolants near $\tilde{E}$. This seems to be the first known example demonstrating singular behavior in non-linear spline interpolation.

§6 contains an exhaustive study of extremal P-interpolants for the case where P consists of two points. It is shown that there exist, besides the trivial extremal, countably many non-trivial ones of distinct integral mode, that all of them are obtained by simple transformations from a basic one, all have the same length and (cf. [5]) none makes the potential energy a local minimum. Composition of these 2-point extremals yields countably many extremal P-interpolants for various special configurations P. §6 also exhibits countably many angle-prescribed and countably many length-prescribed 2-point extremals.

In §7 some special cases of closed extremal P-interpolants are considered. It is shown that the only closed length-prescribed extremal without knots are the repeatedly traversed circle and figure eight configurations. Formally, the Euler equation is the limiting case of the Euler equation for an elastic circular ring under hydrostatic pressure $p$ as $p \to 0$ (cf. [1] and [12]). The ring, however, is not an elastica since its deformations satisfy stress-strain relationships. We also consider closed extremals which are not length-prescribed. Here the extremals in §6 are used to construct infinitely many closed extremals for several special P-configurations, for example where P is the set of vertices of a regular polygon. In particular, if a regular m-gon, $m \geq 2$, is inscribed in the unit circle, then a circumscribed extremal exists with length

$$s_m = \frac{2m \sin\frac{\pi}{m} F(\frac{1}{2}\sqrt{2};\ \beta_m)}{\sqrt{2}[2E(\frac{1}{2}\sqrt{2};\ \beta_m) - F(\frac{1}{2}\sqrt{2};\ \beta_m)]} ,$$

where $\cos\beta_m = \sqrt{\cos \pi/m}$. There are similar formulas for the energy $U_m$ and the arc length $s_m(\theta)$; of interest is the result that $s_m(\theta)/\theta \to 1$ as $m \to \infty$, so that the circumscribed extremals have the unit circle as a limiting configuration. These extremals are stable, i.e. they make the potential energy a local minimum, as proved in [5].

§2. Regularity and Characterizations of Open Extremals

For two points $p = (p^1, p^2)$ and $q = (q^1, q^2)$ in real Euclidean space $\mathbb{R}^2$ we employ the inner product $pq = p^1q^1 + p^2q^2$, the distance $|p-q| = [(p-q)(p-q)]^{1/2}$, and the exterior product $[p,q] = p^1q^2 - p^2q^1$ of such points. We consider mappings $x = (x^1, x^2)$ of the unit interval $I = [0,1]$ to $\mathbb{R}^2$. We denote by $H_2(I)$ the real Hilbert space of those mappings $x$ such that the derivative $\dot{x}$ is absolutely continuous and $\ddot{x} \in L_2(I)$, equipped with the inner product

$$(x,y)_{H_2} = \int_I (xy + \dot{x}\dot{y} + \ddot{x}\ddot{y}) \quad . \tag{2.1}$$

We say $x$ is a regular element of $H_2$ if $|\dot{x}(t)| > 0$ for all $t \in I$. We observe that the regular elements of $H_2$ form an open subset $H_2^{reg}$ of $H_2$.

For $x \in H_2$ we define the arc length map $s_x : I \to \mathbb{R}_+$ by

$$s_x(t) = \int_0^t |\dot{x}|, \quad t \in I \quad . \tag{2.2}$$

If $x \in H_2^{reg}$ then $s_x$ has an inverse $s_x^{-1} : [0,\bar{s}] \to [0,1]$, where $\bar{s} = s_x(1)$, and in this case the function $x \circ s_x^{-1} : [0,\bar{s}] \to \mathbb{R}^2$ has an absolutely continuous derivative, and square-integrable second derivative. We identify $x$ with the oriented curve C in the $x^1x^2$-plane which has parametric representation $x = x(t)$. Writing $\bar{x} = x \circ s_x^{-1}$, we say that $\bar{x}$ is the arc length parametrization of the curve C. Clearly $\bar{x} \in H_2(0,\bar{s})$ and we have:

$$\dot{\bar{x}} = \dot{x} \circ s_x^{-1}/|\dot{x} \circ s_x^{-1}|^2, \quad \ddot{\bar{x}} = \frac{\ddot{x} \circ s_x^{-1}|\dot{x} \circ s_x^{-1}| - \dot{x} \circ s_x^{-1}(\dot{x}\ddot{x} \circ s_x^{-1})}{|\dot{x} \circ s_x^{-1}|^4} \quad .$$

If $x \in H_2^{reg}$, then its curvature $\kappa_x : I \to \mathbb{R}$ is defined by

$$\kappa_x(t) = [\dot{\bar{x}}, \ddot{\bar{x}}] \circ s_x(t) = [\dot{x}, \ddot{x}]|\dot{x}|^{-3}(t), \quad t \in I \quad . \tag{2.3}$$

Suppose $x \in H_2^{reg}$ and $s_x(1) = \bar{s}$. Then we define the <u>curvature functional</u>,

$$U(x) = \int_0^{\bar{s}} \kappa_x^2 = \int_0^{\bar{s}} [\dot{\bar{x}},\ddot{\bar{x}}]^2 = \int_0^{\bar{s}} \ddot{\bar{x}}^2 \quad . \tag{2.4}$$

Note that (2.4) defines $U$ as a mapping of $H_2^{reg}$ into $\mathbb{R}_+$. The equivalent expression,

$$U(x) = \int_I [\dot{x},\ddot{x}]^2|\dot{x}|^{-5} \quad , \tag{2.5}$$

is independent of the parametrization of $x$ in the following sense. If $u$ is a $C^\infty$-map of $I$ onto itself with $\dot{u} > 0$ and $x = y \circ u$ then

$$U(x) = \int_I [\dot{y},\ddot{y}]^2|\dot{y}|^{-5} = U(y) \quad .$$

If $x \in H_2^{reg}$ then $U$ is Fréchet-differentiable at $x$ and, for any increment $y \in H_2$,

$$U'(x)[y] = \int_I \{2[\dot{x},\ddot{x}]([\dot{x},\ddot{y}] + [\dot{y},\ddot{x}])|\dot{x}|^{-5} - 5[\dot{x},\ddot{x}]^2\dot{x}\dot{y}|\dot{x}|^{-7}\} \quad . \tag{2.6}$$

If the variable of integration is chosen to be $s_x$, (2.6) simplifies to

$$U'(x)[y] = \int_0^{\bar{s}} (2\ddot{\bar{x}}\ddot{\bar{y}} - 3\kappa_x^2\dot{\bar{x}}\dot{\bar{y}})ds_x ; \ \bar{x} = x \circ s_x, \ \bar{y} = y \circ s_x \quad . \tag{2.7}$$

Let $0 = p_0,p_1,\ldots,p_m$ be fixed points in $\mathbb{R}^2$, not necessarily distinct, but $p_{i-1} \neq p_i$, $i = 1,\ldots,m$, and let $P$ denote the ordered set $\{p_0,p_1,\ldots,p_m\}$. We refer to $P$ as a <u>configuration</u> *in* $\mathbb{R}^2$. *If* $x \in H_2^{reg}$ *is such that* $x(t_i) = p_i$ $(i = 0,1,\ldots,m)$ for some $0 \le t_0 < t_1 < \cdots < t_m \le 1$ we say the curve $x$ is an <u>admissible P-interpolant</u>, with <u>knots</u> $p_i$ $(i = 0,\ldots,m)$. The <u>terminals</u> $x(0)$, $x(1)$ may or may not be coincident with the terminal knots $p_0,p_m$. The P-interpolants defined here are to be considered as <u>open</u> even if $x(0) = x(1)$. In the physical interpretation $p_i=p_j$ for some $i \neq j$ means that the beam is constrained to pass through two sleeves which are fixed at the same point $p_i$ but can rotate independently of each other.

Suppose $x$ is a fixed admissible P-interpolant and $\bar{x} = x \circ s_x^{-1}$ is its arc length parametrization, $s_x(1) = \bar{s}$ its length, $\bar{x}(\bar{s}_i) = p_i$ $(i = 0,1,\ldots,m)$ its knots. Given any $z \in H_2$ let $\bar{z} = z \circ s_x^{-1}$, be the parametrization of $z$ which uses the arc length of $x$ as the parameter, and assume $\bar{z}(\bar{s}_i) = 0$ $(i = 0,1,\ldots,m)$. For $|\varepsilon|$ sufficiently small, $x + \varepsilon z$ is an admissible P-interpolant and

$$U(x+\varepsilon z) - U(x) = \varepsilon\, U'(x)[z] + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \ . \tag{2.8}$$

This justifies the following

<u>Definition 2.1</u>. The admissible P-interpolant $x$, with arc length parametrization $\bar{x} = x \circ s_x^{-1}$, knots $p_i = \bar{x}(\bar{s}_i)$ $(i = 0,1,\ldots,m)$, length $\bar{s} = s_x(1)$, is an extremal P-interpolant if

$$U'(x)[z] = 0 \ , \tag{2.9}$$

i.e.,

$$\int_0^{\bar{s}} (2\ddot{\bar{x}}\ddot{\bar{z}} - 3\kappa_x^2 \dot{\bar{x}}\dot{\bar{z}}) = 0, \quad \bar{z} = z \circ s_x^{-1} \ ,$$

for every $z \in H_2$ satisfying $\bar{z}(\bar{s}_i) = 0$ $(i = 0,1,\ldots,m)$.

The following proposition follows from (2.8) by the usual arguments of the calculus of variations. It helps to explain the interest in extremal P-interpolants.

<u>Proposition 2.1</u>. Suppose the admissible P-interpolant $x$ minimizes the curvature functional $U$ locally, i.e.

$$U(x) \le U(y)$$

for every admissible P-interpolant $y$ in a neighborhood of $x$ in $H_2$. Then $x$ is an extremal P-interpolant.

The three major propositions of this section follow. We use the notation $y_J$ for the restriction of a map $y$ to the interval $J$.

**Proposition 2.2.** The admissible P-interpolant $x$ with arc length parametrization $\bar{x}$, knots $p_i = \bar{x}(\bar{s}_i)$ $(i = 0,\dots,m)$, and length $\bar{s}$, is extremal if and only if the conditions

$$(2.10)\quad \begin{aligned} &\text{(i)}\quad \bar{x} \in C^2[0,\bar{s}],\ \ddot{\bar{x}}(s) = 0 \quad \text{for} \quad 0 \le s \le \bar{s}_0 \quad \text{and} \quad \bar{s}_m \le s \le \bar{s} \\ &\text{(ii)}\quad (2\dddot{\bar{x}} + 3\kappa_x^2\dot{\bar{x}})(s) = c_i \in \mathbb{R}^2 \quad \text{for} \quad s \in (\bar{s}_{i-1},\bar{s}_i),\ i = 1,\dots,m \end{aligned}$$

hold with $\bar{x}_{(\bar{s}_{i-1},\bar{s}_i)} \in C^\infty(\bar{s}_{i-1},\bar{s}_i)$ $(i = 1,\dots,m)$.

Remark 2.1. Throughout the paper we use the same symbol to denote regularity classes for both scalar and vector functions.

Proof: The implication (2.10) $\Longrightarrow$ (2.9) is routine and follows upon decomposing $[0,\bar{s}]$ into subintervals determined by the $\bar{s}_i$, dot-multiplying (2.10ii) by $\dot{z}$, integrating by parts and summing; the continuity of $\ddot{\bar{x}}\ddot{\bar{z}}$, the equations $\bar{z}(\bar{s}_i) = 0$ $(i = 0,\dots,m)$ and the equations of (2.10) easily yield $U'(x)[z] = 0$.

Conversely, if (2.9) holds then, selecting $\bar{z} \in C^\infty[0,\bar{s}]$ with support in $[\bar{s}_i,\bar{s}_{i+1}]$, $i$ fixed, we have

$$\int_{\bar{s}_i}^{\bar{s}_{i+1}} (2\ddot{\bar{x}} + F)\ddot{\bar{z}} = 0 \quad ,$$

where $\dot{F} = 3\kappa_x^2\dot{\bar{x}}$. By elementary distribution theory, $(2\ddot{\bar{x}} + F)_{(\bar{s}_i,\bar{s}_{i+1})}$ is in $C^\infty(\bar{s}_i,\bar{s}_{i+1})$ and

$$D^2(2\ddot{\bar{x}} + F)_{(\bar{s}_i,\bar{s}_{i+1})} = 0 \quad .$$

It follows that

$$(2\dddot{\bar{x}} + 3\kappa_x^2\dot{\bar{x}})_{(\bar{s}_i,\bar{s}_{i+1})} = c_i$$

and, recursively, $\bar{x}_{(\bar{s}_i,\bar{s}_{i+1})} \in C^\infty(\bar{s}_i,\bar{s}_{i+1})$. To prove the continuity of $\ddot{\bar{x}}$ at an interior knot $\bar{s}_i$, select $u$ in $C^\infty[0,\bar{s}]$ with support in $[\bar{s}_{i-1},\bar{s}_{i+1}]$ satisfying $u(\bar{s}_i) = 0$,

$u'(\bar{s}_i) = 1$ and put $z = (u,0)$. Then, from (2.9) and integration by parts,

$$0 = \int_{\bar{s}_{i-1}}^{\bar{s}_{i+1}} \{(2\dddot{\bar{x}} + 3\kappa_{\bar{x}}^2\dot{\bar{x}})\dot{z}\} + 2(1,0)(\ddot{\bar{x}}(\bar{s}_i + 0) - \ddot{\bar{x}}(\bar{s}_i - 0)) \quad .$$

Since the first term equals

$$c_i(z(\bar{s}_i) - z(\bar{s}_{i-1})) + c_{i+1}(z(\bar{s}_{i+1}) - z(\bar{s}_i)) \quad ,$$

which is clearly zero, we conclude that $(\bar{x}^1)_{(\bar{s}_0,\bar{s}_m)}$ is in $C^2(\bar{s}_0,\bar{s}_m)$. A similar argument works for $\bar{x}^2$.

If $\bar{s}_i$ is either $\bar{s}_0 = 0$ or $\bar{s}_m = \bar{s}$, jumps are replaced by one-sided limits and one concludes $\ddot{\bar{x}}(\bar{s}_0 + 0) = \ddot{\bar{x}}(\bar{s}_m - 0) = 0$. Assume now $\bar{s}_m < \bar{s}$. One argues as above that

$$(2\dddot{\bar{x}} + 3\kappa_{\bar{x}}^2\dot{\bar{x}})_{(\bar{s}_m,\bar{s})} = c_m \quad ,$$

and that $\ddot{\bar{x}}$ is continuous at $\bar{s}_m$. We show that $c_m = 0$. Indeed, select $u \in C^\infty[0,\bar{s}]$, with $u \equiv 0$ for $0 \le s \le \bar{s}_m$, satisfying $u(\bar{s}) = 1$, $\dot{u}(\bar{s}) = 0$, and put $z = (u,0)$. Then, from (2.9) and integration by parts over $[\bar{s}_m,\bar{s}]$,

$$0 = c_m(z(\bar{s}) - z(\bar{s}_m)) = c_m^1$$

and a similar result holds for $c_m^2$. Thus $c_m = 0$. By (2.15i) of Proposition 2.4 to follow, we conclude that $\kappa_{\bar{x}}^2(s) = 0$ for $\bar{s}_m < s < \bar{s}$. In particular, $\ddot{\bar{x}}(s) = 0$ for $\bar{s}_m < s < \bar{s}$, and, by continuity, for $\bar{s}_m \le s \le \bar{s}$. A similar proof holds if $0 < \bar{s}_0$. This completes the proof of the proposition.

We introduce the following notation for the jump of the third derivative of the extremal $\bar{x}$ at the knot $\bar{s}_i$

(2.11) $$\Delta_i\dddot{\bar{x}} = \dddot{\bar{x}}(\bar{s}_i + 0) - \dddot{\bar{x}}(\bar{s}_i - 0), \quad i = 0,1,\ldots,m \quad .$$

If $\dddot{\bar{x}}(\bar{s}_0 - 0)$ and/or $\dddot{\bar{x}}(\bar{s}_m + 0)$ are not defined, they are to be replaced by $0$.

The following proposition gives expressions for the extremal value $U(x)$ which do not involve quadratures.

<u>Corollary 2.1</u>. If $\bar{x}$ is the extremal P-interpolant of Proposition 2.2, then

(2.12i)
$$U(\bar{x}) = \sum_{i=0}^{m-1} c_i (p_{i+1} - p_i) \quad ,$$

and

(2.12ii)
$$U(\bar{x}) = -2 \sum_{i=0}^{m} p_i \Delta_i \dddot{\bar{x}}$$

<u>Proof</u>. If we dot-multiply (2.10ii) by $\dot{\bar{x}}$ and integrate over $[0,\bar{s}]$, using integration by parts, we obtain

$$-2U(\bar{x}) + 3U(\bar{x}) = \sum_{i=0}^{m-1} c_i (p_{i+1} - p_i) \quad ,$$

which is (2.12i). Now use (2.10ii) at $\bar{s}_i + 0$ and $\bar{s}_i - 0$ and subtract to obtain $2\Delta_i \dddot{\bar{x}} = c_i - c_{i-1}$, which holds for $i = 0,1,\ldots,m$ if we define $c_{-1} = c_m = 0$. By (2.12i) we have

$$U(\bar{x}) = \sum_{i=0}^{m} c_i (p_{i+1} - p_i) = -\sum_{i=0}^{m} p_i (c_i - c_{i-1}) = -2 \sum_{i=0}^{m} p_i \Delta_i \dddot{\bar{x}} \quad ,$$

so (2.12ii) is also proved.

<u>Remark 2.2</u>. Since $U(x) = 0$ only if $x$ is linear, it follows from (2.12ii) that an extremal P-interplant $\bar{x}$ that is not linear must have a discontinuity of $\dddot{\bar{x}}$ at some of the knots (or else, $\dddot{\bar{x}}(p_0 + 0) \neq 0$ or $\dddot{\bar{x}}(p_m - 0) \neq 0$).

From an extremal $x$, as characterized in Proposition 2.2, one can obtain infinitely many other extremals by shifting the terminals $x(0)$ and $x(1)$ along the rays that are tangent to $x$ at $p_0$ and $p_m$. The value of $U(x)$ is not changed by these variations. We wish to ignore these trivial portions of an extremal and will for this reason adopt the following convention. If we speak of an extremal P-interpolant $x$ with arc parametrization $\bar{x}$, knots $p_i = \bar{x}(\bar{s}_i)$ $(i = 0,\ldots,m)$ and length $\bar{s}$ then, unless stated otherwise, $\bar{s}_0 = 0$, $\bar{s}_m = \bar{s}$, and the terminals are $p_0 = \bar{x}(0) = 0$, $p_m = \bar{x}(\bar{s}_m) = \bar{x}(\bar{s})$.

For some applications one wishes to constrain the p-interpolants further by prescribing the length $\bar{s}$ of the arc between the terminals. Let the class of these P-interpolants be called <u>length-prescribed</u> (they differ from the "length-constrained" interpolants of [3]). The next definition deals with the extremals for $U$ in this class.

<u>Definition 2.2</u>. The admissible P-interpolant $x$, with arc length parametrization $\bar{x} = x \circ s_x^{-1}$, knots $p_i = \bar{x}(\bar{s}_i)$ $(i = 0,\ldots,m)$, length $\bar{s} = s_x(1)$, is a length-prescribed extremal P-interpolant if

$$U'(x)[z] + \lambda S'(x)[z] = 0 \tag{2.13i}$$

for any $z \in H_2(I)$ for which $z \circ s_x^{-1}(\bar{s}_i) = 0$ $(i = 0,\ldots,m)$ and $\lambda \in \mathbb{R}$ determined so that

$$S(x) := \int_I |\dot{x}| = \bar{s} \quad . \tag{2.13ii}$$

For these extremals we have a characterization similar to that of Proposition 2.2; note that $S'(x)$ is given by

$$S'(x)[y] = \int_0^{\bar{s}} \dot{\bar{x}}\dot{\bar{y}}, \quad \text{for all } y \in H_2(I);\ \bar{x} = x \circ s_x^{-1},\ \bar{y} = y \circ s_x^{-1} \quad .$$

<u>Proposition 2.3</u>. The admissible P-interpolant $x$ with arc length parametrization $\bar{x}$, knots $p_i = \bar{x}(\bar{s}_i)$ $(i = 0,1,\ldots,m)$ and length $\bar{s} = \bar{s}_m$ is a length-prescribed extremal if and only if

$$\text{(i)}\quad \bar{x} \in C^2[0,\bar{s}],\ \ddot{\bar{x}}(0) = 0,\ \bar{x}(\bar{s}) = 0 \quad ,$$

$$\text{(ii)}\quad (2\dddot{\bar{x}} + 3\kappa_x^2\dot{\bar{x}} - \lambda\dot{\bar{x}})_{(\bar{s}_{i-1},\bar{s}_i)} = c_i \in \mathbb{R}^2,\ i = 1,\ldots,m \quad , \tag{2.14}$$

$$\text{(iii)}\quad \lambda\bar{s} = U(\bar{x}) - \sum_{i=1}^{m} c_i(p_i - p_{i-1}) = U(\bar{x}) + 2\sum_{i=0}^{m} p_i\Delta_i\dddot{\bar{x}} \quad ,$$

hold with $\bar{x}_{(\bar{s}_{i-1},\bar{s}_i)} \in C^\infty(\bar{s}_{i-1},\bar{s}_i)$ $(i = 1,\ldots,m)$ .

Proof. To prove the implication (2.14i,ii) $\Longrightarrow$ (2.13i) one proceeds as in the first part of the proof of Proposition 2.2. If, next, (2.14ii) is dot-multiplied by $\dot{\bar{x}}$ and integrated over $[0,\bar{s}]$ one obtains, using integration by parts,

$$-2U(\bar{x}) + 3U(\bar{x}) - \lambda\bar{s} = \sum_{i=1}^{m} c_i(p_i - p_{i-1}) = -2\sum_{i=0}^{m} p_i\Delta_i\dddot{\bar{x}}$$

which give (2.14iii), which is seen to be equivalent to (2.13ii).

Conversely, if (2.13i) holds then selecting $z \in C^\infty[0,\bar{s}]$ with support in $[\bar{s}_i,\bar{s}_{i+1}]$, $i$ fixed, we have

$$\int_{\bar{s}_i}^{\bar{s}_{i+1}} (2\ddot{\bar{x}} + F - \lambda\dot{\bar{x}})\ddot{\bar{z}} = 0$$

where $F$ is as in the proof of Proposition 2.2. It follows that

$$(2\dddot{\bar{x}} + 3\kappa_{\bar{x}}^2\dot{\bar{x}} - \lambda\dot{\bar{x}})_{(\bar{s}_i,\bar{s}_{i+1})} = c_i \quad .$$

The regularity properties of $x$ are proved as before. The argument in the first part of the proof shows, that given (2.14i,ii), then (2.14iii) and (2.13ii) are equivalent.

Remark 2.3. For any configuration $P = \{p_0,p_1,\ldots,p_m\}$, there exists an extremal P-interpolant satisfying (2.14) with $\lambda \in \mathbb{R}$. Indeed, if

$$L_0 = \sum_{i=0}^{m-1} |p_{i+1} - p_i| \quad ,$$

then the length-prescribed extremal $x$, which minimizes $\int_0^L \kappa_x^2$ among all admissible P-interpolants with length equal to $L > L_0$ is guaranteed to exist [6] and satisfies (2.13i) [cf. Appendix].

Remark 2.4. Curves $\bar{x} : \mathbb{R} \to \mathbb{R}^2$, satisfying the equation

$$2\dddot{\bar{x}} + 3\kappa_x^2\dot{\bar{x}} - \lambda\ddot{\bar{x}} = c$$

are called elastica (cf. [8]), more specifically inflexional elastica if the curve has inflection points (which is the case if and only if $\lambda^2 \leq c^2$). Curves for which $\lambda = 0$, the case of primary interest in this paper, will be referred to as simple elastica. Geometrically, simple elastica are characterized by the property that the angular variation between consecutive inflection points is exactly $\pi$ (for all inflexional elastica the angular variation is $\geq \pi$). A smooth oriented curve in $\mathbb{R}^2$ with continuous curvature which consists of finitely many subarcs of the simple elastica and has (possibly) discontinuities of the curvature derivative at the interpolation points $p_1,\ldots,p_{m-1}$ only is called an interpolating elastica (cf. [8]).

The next proposition deals with implications and equivalences of (2.14ii). It should be observed that these results apply to Equation (2.10ii) as well, since the latter is the special case of (2.14ii) with $\lambda = 0$.

Proposition 2.4. Condition (2.14ii) implies each of the following four conditions on $(\bar{s}_{i-1},\bar{s}_i)$ $(i = 1,\ldots,m)$:

(2.15)

$$\text{(i)} \quad \kappa_x^2 = c_i\dot{\bar{x}} + \lambda \quad ,$$

$$\text{(ii)} \quad \dot{\kappa}_x = \tfrac{1}{2}[\dot{\bar{x}},\, c_i] \quad ,$$

$$\text{(iii)} \quad \kappa_x = \tfrac{1}{2}[\bar{x},\, c_i] + \gamma_i,\ \gamma_i \in \mathbb{R} \quad ,$$

$$\text{(iv)} \quad \ddot{\kappa}_x + \tfrac{1}{2}\kappa_x^3 - (\lambda/2)\kappa_x = 0 \quad .$$

Moreover, (2.15i) also implies (2.14ii).

Proof: If (2.14ii) is dot-multiplied by $\dot{\bar{x}}$, one obtains, since $|\dot{\bar{x}}| = 1$, $|\ddot{\bar{x}}|^2 = \kappa_x^2$, $\dot{\bar{x}}\ddot{\bar{x}} = 0$, $\dot{\bar{x}}\dddot{\bar{x}} + |\ddot{\bar{x}}|^2 = 0$:

$$3\kappa_x^2 + 2\dot{\bar{x}}\dddot{\bar{x}} = \kappa_x^2 + 2|\ddot{\bar{x}}|^2 + 2\dot{\bar{x}}\dddot{\bar{x}} = \kappa_x^2 = c_i\dot{\bar{x}} + \lambda \quad ,$$

hence (2.15i). Differentiating (2.15i) we obtain

$$\kappa_x \dot{\kappa}_x = \frac{1}{2} c_i \ddot{\bar{x}} \quad .$$

If $\kappa_x(s) \neq 0$ for some $s$, then $|\kappa_x(s)| = |\ddot{\bar{x}}(s)| > 0$ and $\ddot{\bar{x}}(s)/\kappa_x(s)$ is the unit vector $(-\dot{\bar{x}}^2(s), \dot{\bar{x}}^1(s))$. Therefore, $\dot{\kappa}_x(s) = \frac{1}{2}[\dot{\bar{x}}(s), c_i]$ and (2.15ii) holds in this case. If $\kappa_x(s) = 0$[†] and $s$ is a limit of $s_n$ such that $\kappa_x(s_n) \neq 0$, then continuity of $\dot{\bar{x}}$ and $\dot{\kappa}_x$ together with the previous argument give the same equation. On a fixed interval $(\bar{s}_{i-1}, \bar{s}_i)$, let $\Gamma_i = \{s : \kappa_x(s) \neq 0\}$ and let $(\alpha,\beta)$ be any subinterval in the decomposition of $(\bar{s}_{i-1}, \bar{s}_i)\setminus\bar{\Gamma}_i$. We must show that (2.15ii) holds on $(\alpha,\beta)$. Now $\kappa_x$ and $\dot{\kappa}_x$ are zero on $(\alpha,\beta)$. By the continuity of $\dot{\kappa}_x$ on $(\bar{s}_{i-1}, \bar{s}_i)$ it follows that $\dot{\kappa}_x(\beta) = 0 = [\dot{\bar{x}}(\beta), c_i]$. Moreover, since $\kappa_x = 0$ on $(\alpha,\beta)$,

$$\bar{x}(s) = as + b,\ s \in [\alpha,\beta],\ a,b \in \mathbb{R}^2 \quad . \tag{2.16}$$

Thus, from (2.16),

$$[\bar{x}(s), c_i] = [a, c_i],\ s \in [\alpha,\beta] \quad .$$

In particular, $[a, c_i] = 0$ and thus (2.15ii) holds on $(\alpha,\beta)$. Next, (2.15iii) follows from (2.15ii) by integration. To show that (2.15iv) holds, let $\kappa_x(s) \neq 0$; then $\ddot{\bar{x}}(s) = \kappa_x(s)(-\dot{\bar{x}}^2(s), \dot{\bar{x}}^1(s))$ and one obtains upon differentiating (2.15ii),

$$\ddot{\kappa}_x(s) = \frac{1}{2}[\ddot{\bar{x}}(s), c_i] = -\frac{1}{2}\kappa_x(s) c_i \dot{\bar{x}}(s) = \frac{1}{2}[\lambda\kappa_x(s) - \kappa_x^3(s)] \quad .$$

Thus, (2.15iv) holds at every point $s$ for which $\kappa_x(s) \neq 0$. The case when $s$ is a limit point of $s_n$ for which $\kappa_x(s_n) \neq 0$ then follows immediately. The case when $s$ is not such a limit point is of course trivial.

To show that (2.15i) implies (2.14ii), assume $\ddot{\bar{x}}(s) \neq 0$ for some $s \in (\bar{s}_i, \bar{s}_{i+1})$. Since $\dot{\bar{x}}(s)$ and $\ddot{\bar{x}}(s)/|\ddot{\bar{x}}(s)|$ are orthogonal unit vectors in $\mathbb{R}^2$, we have, using (2.15i) and the fact that $\dot{\bar{x}}(s)\dddot{\bar{x}}(s) + |\ddot{\bar{x}}(s)|^2 = 0$,

---

[†] The authors thank S. D. Fisher for a helpful suggestion here.

$$c_i = (c_i \dot{\bar{x}}(s))\hat{\bar{x}}(s) + (c_i \ddot{\bar{x}}(s))\ddot{\bar{x}}(s)/|\ddot{\bar{x}}(s)|^2$$

$$= (\kappa_x^2(s) - \lambda)\dot{\bar{x}}(s) + \frac{d(\kappa_x^2(s))}{ds}\ddot{\bar{x}}(s)/|\ddot{\bar{x}}(s)|^2$$

$$= (3\kappa_x^2(s) - \lambda)\dot{\bar{x}}(s) + (2\dddot{\bar{x}}(s)\dot{\bar{x}}(s))\dot{\bar{x}}(s) + (2\dddot{\bar{x}}(s)\ddot{\bar{x}}(s))\ddot{\bar{x}}(s)/|\ddot{\bar{x}}(s)|^2$$

$$= (3\kappa_x^2(s) - \lambda)\dot{\bar{x}}(s) + 2\dddot{\bar{x}}(s) \quad ,$$

which is just (2.14ii). If $s$ is a limit point of $s_n$ for which $\ddot{\bar{x}}(s_n) \neq 0$, then the same equation holds by continuity. If $s$ is not such a limit point, then (2.14ii) reduces to $c_i = -\lambda\dot{\bar{x}}(s)$, which is clearly obtained by dot-multiplication of (2.15i) by $\dot{\bar{x}}(s)$. This concludes the proof of Proposition 2.4.

At the end of this section we mention still another constraining condition for extremal interpolants. It consists in fixing the angle that the interpolant $x$ makes with a fixed line at the terminal knot $p_0 = \bar{x}(\bar{s}_0)$ and/or $p_m = \bar{x}(\bar{s}_m)$. Thus, the condition is

$$\dot{\bar{x}}(\bar{s}_0) = e_0 \quad \text{and/or} \quad \dot{\bar{x}}(\bar{s}_m) = e_m \tag{2.17}$$

where $e_0, e_m$ are unit vectors in $\mathbb{R}^2$. We refer to these extremals as angle-constrained.

If $x$ is an admissible P-interpolant with knots $p_i = \bar{x}(\bar{s}_i)$ which satisfies the constraint (2.17) then any other P-interpolant in a sufficiently small $H_2$-neighborhood of $x$, satisfying the same condition, is of the form $x + \varepsilon z$ where $z \in H_2$, $z \circ s_x^{-1}(\bar{s}_i) = 0$ $(i = 0,1,\ldots,m)$, $\dot{z} \circ s_x^{-1}(\bar{s}_0) = 0$ and/or $\dot{z} \circ s_x^{-1}(\bar{s}_m) = 0$. It is easily seen that if $\bar{x}$ is an extremal P-interpolant with the added constraint $\dot{\bar{x}}(\bar{s}_0) = e_0$ then the "free boundary" condition $\ddot{\bar{x}}(\bar{s}_0) = 0$ of (2.10i) is replaced by $\dot{\bar{x}}(\bar{s}_0) = e_0$. Similarly if $\dot{\bar{x}}(\bar{s}_m) = e_m$ is a constraint then this condition replaces $\ddot{\bar{x}}(\bar{s}_m) = 0$. There is no other change in the conditions of Proposition 2.2.

§3. Normal Representation of Extremals

In this section the dependent variable is an angle. Let $T(=T^1)$ denote the 1-dimensional torus. $\phi \in T$ is represented by a real number (also denoted as) $\phi$, one of the set $\phi + 2k\pi (k = 0, \pm 1, \pm 2, \ldots)$. A continuous function $\theta : (0,\bar{s}) \to T$ is represented by a continuous function (also denoted as) $\theta : (0,\bar{s}) \to \mathbb{R}$, one of the set $\theta + 2k\pi$. The derivative $\dot{\theta}$ is always a unique function $(0,\bar{s}) \to \mathbb{R}$. Let $\mathring{H}_1(0,\bar{s})$ denote the class of absolutely continuous functions $\theta : (0,\bar{s}) \to \mathbb{R}$ for which $\dot{\theta} \in L_2(0,\bar{s})$. Then the function $x = x_\theta :$ $(0,\bar{s}) \to \mathbb{R}^2$, defined by

$$x_\theta^1(s) = \int_0^s \cos\theta, \quad x_\theta^2(s) = \int_0^s \sin\theta \tag{3.1}$$

is in $H_2(0,\bar{s})$ and represents an oriented curve $C$ in the $x^1x^2$-plane, parametrized with respect to arc length, $x(0) = 0$, $\dot{x}_2/\dot{x}_1 = \tan\theta$, $\theta(s)$ is the angle which the curve $C$ makes with the $x^1$-axis at arc length $s$, and $\dot{\theta}_s$ is the curvature of $C$ at $s$. Conversely, given an oriented curve $C$ with *cartesian representation* $x \in H_2(0,\bar{s})$, parametrized with respect to arc length, there is a unique function $\theta_x \in \mathring{H}_1(0,\bar{s})$ such that

$$x^1(s) = x^1(0) + \int_0^s \cos\theta_x, \quad x^2(s) = x^2(0) + \int_0^s \sin\theta_x \quad .$$

We say that $\theta_x$ is the normal representation (n.r.) of $C$.

Curves $C$ that differ by a translation have the same normal representation. If $\theta_x$ is the n.r. of $C$ then $\theta_x$ + const. is the n.r. of a curve obtained from $C$ by a rotation, and $-\theta_x$ is the n.r. of a curve obtained from $C$ by a reflection at the $x^1$-axis. In a geometric setting we may identify curves $C$ which differ only by a congruence, and each congruence class is represented by a single function $\theta$ with the specification $\theta(0) = 0$, $\dot{\theta}(0) > 0$ (or $\ddot{\theta}(0) > 0$ if $\dot{\theta}(0) = 0$).

In many cases it will be convenient to characterize extremal P-interpolants by their normal representation. For this purpose we replace Propositions 2.2 and 2.3 by the following propositions whose proofs we omit.

**Proposition 3.1.** The function $\bar{\theta} \in \mathring{H}_1(0,\bar{s})$ is the normal representation of an extremal P-interpolant with knots $p_i$ $(i = 0,1,\ldots,m)$ at $0 = \bar{s}_0 < \bar{s}_1 < \cdots < \bar{s}_m$ and length $\bar{s} = \bar{s}_m$, if and only if the conditions

$$
\begin{aligned}
&\text{(i)} \quad \bar{\theta} \in C^1[0,\bar{s}],\ \dot{\bar{\theta}}(0),\ \dot{\bar{\theta}}(\bar{s}) = 0 \ , \\
&\text{(ii)} \quad \dot{\bar{\theta}}^2(s) = c_i^1 \cos \bar{\theta}(s) + c_i^2 \sin \bar{\theta}(s) \quad \text{for } s \quad (\bar{s}_{i-1},\bar{s}_i) \ , \\
&\qquad\qquad c_i \in \mathbb{R}^2,\ i = 1,\ldots,m \ , \\
&\text{(iii)} \quad \int_0^{\bar{s}_i} \cos \bar{\theta}(s)ds = p_i^1,\ \int_0^{\bar{s}_i} \sin \bar{\theta}(s)ds = p_i^2,\ i = 1,\ldots,m \ ,
\end{aligned}
\tag{3.3}
$$

hold with $\bar{\theta}_{(\bar{s}_{i-1},\bar{s}_i)} \in C^\infty(\bar{s}_{i-1},\bar{s}_i)$ $(i = 1,\ldots,m)$.

**Proposition 3.2.** The function $\bar{\theta} \in \mathring{H}_1(0,\bar{s})$ is the normal representation of a length-prescribed extremal P-interpolant with knots $p_i$ $(i = 0,1,\ldots,m)$ at $0 = \bar{s}_0 < \bar{s}_2 < \cdots < \bar{s}_m$ and length $\bar{s} = \bar{s}_m$ if and only if the conditions

$$
\begin{aligned}
&\text{(i)} \quad \bar{\theta} \in C^1[0,\bar{s}],\ \bar{\theta}(0) = \dot{\bar{\theta}}(\bar{s}) = 0 \ , \\
&\text{(ii)} \quad \dot{\bar{\theta}}^2(s) = c_i^1 \cos \bar{\theta}(s) + c_i^2 \sin \bar{\theta}(s) + \lambda \quad \text{for } s \in (\bar{s}_{i-1},\bar{s}_i) \ , \\
&\qquad\qquad c_i \in \mathbb{R}^2,\quad i = 1,\ldots,m \ , \\
&\text{(iii)} \quad \int_0^{\bar{s}_i} \cos \bar{\theta}(s)ds = p_i^1,\ \int_0^{\bar{s}_i} \sin \bar{\theta}(s)ds = p_i^2,\ i = 1,\ldots,m \ , \\
&\text{(iv)} \quad \lambda\bar{s} = \int_0^{\bar{s}} \dot{\bar{\theta}}^2 - \sum_{i=1}^m c_i(p_i - p_{i-1}) \ ,
\end{aligned}
\tag{3.6}
$$

hold with $\bar{\theta}_{(\bar{s}_{i-1},\bar{s}_i)} \in C^\infty(\bar{s}_{i-1},\bar{s}_i)$ $(i = 1,\ldots,m)$ .

**Remark 3.1.** The conditions $\dot{\bar\theta}(\bar s_i - 0) = \dot{\bar\theta}(\bar s_i + 0)$ $(i = 1,\ldots,m-1)$, $\dot{\bar\theta}(0) = 0$, and $\dot{\bar\theta}(\bar s_m) = 0$ result in $m + 1$ conditions on the vector constants $c_1,\ldots,c_m$. Both in Proposition 3.1 and 3.2 we have

$$
(3.7)\quad
\begin{aligned}
&\text{(i)}\quad (c^1_{i+1}-c^1_i)\cos\bar\theta(\bar s_i) + (c^2_{i+1}-c^2_i)\sin\bar\theta(\bar s_i) = 0 \;, \\
&\qquad\qquad\qquad\qquad i = 1,\ldots,m-1 \\
&\text{(ii)}\quad c^1_1 \cos\bar\theta(0) + c^2_1 \sin\bar\theta(0) + \lambda = 0 \;, \\
&\text{(iii)}\quad c^1_m \cos\bar\theta(\bar s_m) + c^2_m \sin\bar\theta(\bar s_m) + \lambda = 0 \;,
\end{aligned}
$$

where $\lambda$ is to be taken as $0$ in the case of Proposition 3.1.

For use in Sections 4 and 5 we state and prove

**Proposition 3.3.** The function $\bar\theta \in \mathring{H}_1(0,\bar s)$ is the normal representation of an extremal P-interpolant with knots $p_i$ $(i = 0,\ldots,m)$ *at* $0 = \bar s_0 < \cdots < \bar s_m$ *and length* $\bar s = \bar s_m$ if and only if the conditions

$$
(3.8)\quad
\begin{aligned}
&\text{(i)}\quad 2\ddot{\bar\theta}(s) + c^1_i \sin\bar\theta(s) - c^2_i \cos\bar\theta(s) = 0 \quad \text{for} \\
&\qquad\qquad s \in (\bar s_{i-1},\bar s_i),\; c_i \in \mathbb{R}^2 \;. \\
&\text{(ii)}\quad (c^1_{i+1}-c^1_i)\cos\bar\theta(\bar s_i) + (c^2_{i+1}-c^2_i)\sin\bar\theta(\bar s_i) = 0 \;, \\
&\qquad\qquad\qquad\qquad i = 1,\ldots,m \;, \\
&\text{(iii)}\quad \dot{\bar\theta}(0) = 0,\; \dot{\bar\theta}(\bar s) = 0 \;,
\end{aligned}
$$

hold with $\bar\theta_{(\bar s_{i-1},\bar s_i)} \in C^\infty(\bar s_{i-1},\bar s_i)$ $(i = 1,\ldots,m)$ and $c_{m+1} = 0$.

**Proof.** The forward implication follows directly from Propositions 2.3 and 2.4. The converse implication follows upon multiplying (3.8i) by $\dot{\bar\theta}(s)$ and integrating; if the integrated equation is evaluated at $s = \bar s_m$ and (3.8ii,iii) is used, the constant of integration is seen to be $0$. Thus, (3.8) implies

$$\dot{\bar{\theta}}^2(s) - c_i^1 \cos \bar{\theta}(s) - c_i^2 \sin \bar{\theta}(s) = 0 \quad \text{for} \quad s \in (\bar{s}_{i-1}, \bar{s}_i) \quad ,$$

$$\bar{\theta} \in C^1[0,\bar{s}], \ i = 1,\ldots,m \quad ,$$

and the result now follows from Proposition 3.1.

## §4. Manifold of Extremals

As noted in the previous section, we may consider equivalence classes of curves differing by a congruence, with representer satisfying $x(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) > 0$; or $\ddot{\theta}(0) > 0$ if $\dot{\theta}(0) = 0$.

**Definition 4.1.** The extremal interpolant $E$ is proper if:

(4.1)

(i) $E$ has nonzero curvature $\kappa_i$ at each internal interpolation node $s_i$, $i = 1,\ldots,m-1$.

(ii) Each internal interpolation node is a genuine knot, i.e., there is a discontinuity $\Delta_i \dot{\kappa} \neq 0$ in the derivative of the curvature at $s_i$, $i = 1,\ldots,m-1$.

**Definition 4.2.** The m-tuple $(k_1,\ldots,k_m)$ of nonnegative integers is the mode of $E$ if there are $k_j$ inflection points strictly between the $(j-1)$th and $j$th interpolation nodes; here, an inflection point denotes a point of zero curvature and we note that $\dot{\theta}$ must change sign. For a fixed mode $(k_1,\ldots,k_m)$, $\mathcal{E} = \mathcal{E}_{(k_1,\ldots,k_m)}$ will denote the class of proper $(m+1)$-extremal interpolants $E$ in the mode $(k_1,\ldots,k_m)$.

**Proposition 4.1.** $\mathcal{E}$ is a (finite-dimensional) metric space under the metric

$$d(E_1,E_2) = \max_{0 \le t \le 1} \left|\frac{d}{dt}\, x_1(s_{E_1} t) - \frac{d}{dt}\, x_2(s_{E_2} t)\right| . \tag{4.2}$$

Here $s_{E_j}$ and $x_j$ represent the lengths and Cartesian representations (parametrized w.r.t. arc length) of $E_j$, $j = 1,2$, respectively, where $x_j(0) = 0$, $j = 1,2$.

**Remark 4.1.** We omit the routine proof. We observe that $\mathcal{E}$ is not complete. Indeed, if each $\mathcal{E}_{(k_1,\ldots,k_m)}$ (m fixed) is embedded in the space of all extremal interpolants, with metric described by (4.2), then the boundary $\partial\mathcal{E}$ of $\mathcal{E}$ may contain an extremal interpolant which is not proper. We also observe that if $E_1$ is close to $E_2$, then the configuration interpolated by $E_1$ is close to that interpolated by $E_2$. This would not be true if $E_1$, $E_2$ were not restricted to a class $\mathcal{E}_{(k_1,\ldots,k_m)}$ (see Example 4.2 at the end of this section).

Now let $\theta_E \in C^1[0,s_E]$ be the normal representation of some $E \in \mathcal{E}$. If $0 = s_0 < s_1 < \cdots < s_m = s_E$ are the interpolation nodes of $E$, put $\theta_E(s_i) = \alpha_i$, $i = 0,\ldots,m$ and $\alpha_E = (\alpha_1,\ldots,\alpha_{m-1}) \in T^{m-1}$. Setting $c_i = -2\mu_i$ we have from Propositions 3.1 and 3.3 the existence of a unique multiplier $\mu_E = (\mu_1,\ldots,\mu_m) \in (\mathbb{R}^2)^m$ such that, for $k = 1,\ldots,m$,

$$\text{(i)}\quad \frac{1}{2}\dot\theta_E^2(s) + \mu_k^1 \cos\theta_E(s) + \mu_k^2 \sin\theta_E(s) = 0, \quad s_{k-1} < s < s_k \;,$$

$$\text{(4.3)}\qquad \text{(ii)}\quad \ddot\theta_E(s) - \mu_k^1 \sin\theta_E(s) + \mu_k^2 \cos\theta_E(s) = 0, \quad s_{k-1} < s < s_k \;;$$

and,

$$\text{(iii)}\quad \dot\theta_E(0) = 0, \quad \dot\theta_E(s_E) = 0 \;.$$

By introducing the more convenient notation

$$B_k = (A_k,\beta_k),\ A_k > 0,\ \beta_k \in T^1,\ k = 1,\ldots,m \;,$$

where

$$\mu_k^1 = -A_k \sin\beta_k,\ \mu_k^2 = A_k \cos\beta_k \;,$$

we may rewrite (4.3) in terms of the multipliers $B_k$ for $k = 1,\ldots,m$:

$$\text{(i)}\quad \frac{1}{2}\dot\theta_E^2(s) + A_k \sin(\theta_E(s) - \beta_k) = 0, \quad s_{k-1} < s < s_k \;,$$

$$\text{(4.4)}\qquad \text{(ii)}\quad \ddot\theta_E(s) + A_k \cos(\theta_E(s) - \beta_k) = 0, \quad s_{k-1} < s < s_k \;;$$

and,

$$\text{(iii)}\quad \dot\theta_E(0) = 0,\ \dot\theta_E(s_E) = 0 \;.$$

Since $\theta = 0$ is not a proper extremal we must have $\ddot\theta_E(0+) \neq 0$. We consider it as part of the definition of $\mathcal{E}$ that

$$(4.4\text{iv}) \qquad \ddot{\theta}_E(0+) > 0$$

for all $E \in \mathcal{E}$. Geometrically speaking, $\mathcal{E}$ contains only extremals which turn counterclockwise near the initial point.

Proposition 4.2. $\theta_E \in C^1[0,s_E]$ is the n.r. of an extremal $E \in \mathcal{E}_{(k_1,\ldots,k_m)}$ with interpolation nodes at $0 = s_0 < s_1 < \cdots < s_m = s_E$ if and only if

A. $\theta_E$ satisfies (4.4i - iv) for some $A_k > 0$, $\beta_k \in T^1$.

B. $\sin(\beta_i - \beta_{i+1}) \neq 0$

C. $\ddot{\theta}_E(0+) > 0$, $\operatorname{sgn} \dot{\theta}_E(s_i) = (-1)^{k_1+\cdots+k_i}$ $i = 1,\ldots,m-1$.

Proof: Since $\theta_E \in C^1$ we have, by (4.4i)

$$(4.5\text{i}) \qquad A_i \sin(\alpha_i-\beta_i) - A_{i+1}\sin(\alpha_i-\beta_{i+1}) = 0, \quad i = 1,\ldots,m-1 \quad ,$$

where $\alpha_i = \theta_E(s_i)$. $\quad$ . $\ddot{\theta}_E(s_i-0) = \ddot{\theta}_E(s_i+0)$ if and only if

$$(4.5\text{ii}) \qquad A_i \cos(\alpha_i-\beta_i) - A_{i+1}\cos(\alpha_i-\beta_{i+1}) = 0 \quad .$$

The two equations (4.5i,ii) are equivalent to (4.5i) and $\sin(\beta_i-\beta_{i+1}) = 0$. Thus, the above condition B. is equivalent to the condition that each interpolation node of E be a genuine knot (see 4.1ii). There are $k_i$ inflection points of E between the (i-1)th and i-th interpolation nodes if and only if $\dot{\theta}_E$ changes sign $k_i$ times between $s_{i-1}$ and $s_i$, i.e.

$$(4.6) \qquad \operatorname{sgn} \kappa_{i-1} \cdot \operatorname{sgn} \kappa_i = (-1)^{k_i}, \quad \kappa_i = \dot{\theta}_E(s_i) \quad .$$

Since by (4.4iv) $\dot{\theta}_E(s) > 0$ for all sufficiently small s, (4.6) is equivalent to the above condition C. Condition C. also implies that $\kappa_i \neq 0$ for $i = 1,\ldots,m-1$, thus condition (4.1i) is also satisfied.

<u>Remark 4.2</u>. Condition B. also implies

(4.7) $$A_i - A_{i+1} \neq 0, \quad i = 1,\dots,m-1 \quad .$$

Indeed if $A_i - A_{i+1} = 0$ then by (4.5i) $\sin(\alpha_i-\beta_i) = \sin(\alpha_i-\beta_{i+1})$, hence $\beta_i = \beta_{i+1}$ or $\beta_i = \beta_{i+1} + \pi \pmod{2\pi}$, which contradicts B.

We now define

$$\tilde{\mathbb{R}}^m_+ = \{A \in \mathbb{R}^m_+ : A_i - A_{i+1} \neq 0, \quad i = 1,\dots,m-1\}$$

(4.8) $$\tilde{T}^m = \{\beta \in T^m : \sin(\beta_i-\beta_{i+1}) \neq 0, \quad i = 1,\dots,m-1\}$$

$$B = B_E = (B_1,\dots,B_m), \; \alpha = \alpha_E = (\alpha_1,\dots,\alpha_m) \quad .$$

By Proposition 4.2 each extremal $E \in \mathcal{E}_{(k_1,\dots,k_m)}$ determines a unique point $B_E \in \tilde{\mathbb{R}}^m_+ \times \tilde{T}^m$. In the next three propositions we shall describe the mapping $E \to B_E$ via the composition of two mappings: the homeomorphism

(4.9i) $$J : E \to (\alpha_E, B_E)$$

of $\mathcal{E}$ onto $J(\mathcal{E}) \subset T^{m-1} \times (\tilde{\mathbb{R}}^m_+ \times \tilde{T}^m)$; and the projection

(4.9ii) $$M : (\alpha, B) \to B$$

of $J(\mathcal{E})$ into $\tilde{\mathbb{R}}^m_+ \times \tilde{T}^m$, which is a local diffeomorphism. The composition $M \circ J$ is a global homeomorphism.

<u>Proposition 4.3</u>. The mapping $J$ is a homeomorphism of $\mathcal{E}_{(k_1,\dots,k_m)}$ onto its image.

<u>Proof</u>: The continuity of $J$ follows directly from (4.2) and (4.4); note that $B_1,\dots,B_m$ can be expressed via (4.4) in terms of $\alpha_i$, $\kappa_i$, $\Delta_i\dot{\kappa}$, thus also in terms of $\theta_E = \arctan \dot{x}_{E,2}/\dot{x}_{E,1}$. Suppose now that $J(E) = (\alpha_E, B_E)$. We show that $E \in \mathcal{E}$ is uniquely and continuously determined by its map $(\alpha_E, B_E)$. By (4.4)

$$\sin(\theta_E(0) - \beta_1) = 0, \; \cos(\theta_E(0) - \beta_1) < 0 \quad ,$$

hence $\alpha_0 = \theta_E(0) = \beta_1 + \pi \pmod{2\pi}$. The restriction of $\theta_E$ to $[s_0,s_1]$ is now uniquely determined from

$$\text{(i)} \quad \dot{\theta}_E(s) = [-2A_1 \sin(\theta_E(s) - \beta_1)]^{\frac{1}{2}}$$

$$\text{(4.10)} \qquad \text{(ii)} \quad \theta_E(0) = \alpha_0 \quad ,$$

with $s_1$ uniquely determined from

$$\text{(iii)} \quad \theta_E(s_1) = \alpha_1, \ \dot{\theta}_E(s) = 0 \ \text{ for } k_1 \text{ values of } s \text{ in } (0,s_1) \quad .$$

Indeed if there is an $s_1' > s_1$ for which $\theta_E(s_1) = \theta_E(s_1') = \alpha_1$ then $\dot{\theta}_E(\bar{s}) = 0$ for some $s_1 < \bar{s} < s_1'$, hence $\dot{\theta}_E(s) = 0$ for more than $k_1$ values of $s$ in $(0,s_1')$. This clearly leads to an inductive process; indeed if $\theta_E$ is defined on $[0,s_i]$, one obtains the restriction of $\theta_E$ to $[s_i,s_{i+1}]$ from the initial value problem defined by (4.4ii) with initial values $\theta_E(s_i)$ and $\dot{\theta}_E(s_i)$. $s_{i+1}$ *is uniquely determined from*

$$\theta_E(s_{i+1}) = \alpha_{i+1}, \ \dot{\theta}_E(s) = 0 \ \text{ for } k_{i+1} \text{ values of } s \text{ in } (s_i,s_{i+1}) \quad .$$

The process is terminated at $i = m-1$ by replacing the condition $\theta_E(s_{i+1}) = \alpha_{i+1}$ by $\dot{\theta}_E(s_m) = 0$. Since the continuity of $J^{-1}$ is an easy consequence of (4.4) the proof is complete.

We determine now the image set

$$\text{(4.11)} \qquad S = S_{(k_1,\ldots,k_m)} = J(E_{(k_1,\ldots,k_m)}) \quad .$$

The following are necessary conditions for $(\alpha,B) \in S$:

$$\text{(i)} \quad \sin(\alpha_i - \beta_i) < 0 \quad i = 1,\ldots,m-1 \quad .$$

$$\text{(4.12)} \quad \text{(ii)} \quad \text{If } k_\ell = 0 \text{ for some } 2 \le \ell \le m-1 \text{ then } (-1)^{k_1+\ldots+k_{\ell-1}}\sin(\alpha_\ell - \alpha_{\ell-1}) > 0 \quad .$$

$$\text{(iii)} \quad A_i \sin(\alpha_i - \beta_i) = A_{i+1} \sin(\alpha_i - \beta_{i+1}), \quad i = 1,\ldots,m-1 \quad .$$

Conditions (4.9i and iii) express that $\dot{\theta}_E^2$ is positive and continuous at $s_1,\ldots,s_{m-1}$. If some $k_\ell = 0$ then there must be no inflection point between $s_{\ell-1}$ and $s_\ell$, hence $\sin(\theta_E - \alpha_{\ell-1})$ does not change sign, or

$$\sin(\theta_E(s) - \alpha_{\ell-1}) \cdot \dot{\theta}_E(s_{\ell-1}) > 0 \quad \text{for} \quad s_{\ell-1} < s \leq s_\ell \quad .$$

Using C. of Proposition 4.2, we obtain (4.12ii).

We now show that conditions (4.12) characterize the image set $S$ completely. We observe that (4.12i,ii) define an open set in the $(3m-1)$-dimensional space $T^{m-1} \times (\tilde{\mathbb{R}}_+^m \times \tilde{T}^m)$, while equations (4.9iii) single out a $2m$-dimensional surface in the open set.

<u>Proposition 4.4</u>. The image set $J(E_{(k_1,\ldots,k_m)}$ is

$$S_{(k_1,\ldots,k_m)} = \{(\alpha,B) \in T^{m-1} \times (\tilde{\mathbb{R}}_+^m \times \tilde{T}^m) : \text{conditions (4.12i,ii,iii) hold}\} \quad .$$

<u>Proof</u>: We need to show that if $(\alpha,B)$ *is such that (4.12) holds, then there are numbers*

$$0 = s_0 < s_1 < \cdots < s_m = s_E \quad \text{and a function} \quad \theta_E \in C^1[0,s_E]$$

such that conditions A., C. of Proposition 4.2 are satisfied and moreover,

$$\theta_E(s_i) = \alpha_i, \quad i = 1,\ldots,m-1 \tag{4.13}$$

(condition B. follows from the definition of $\tilde{T}^m$). $s_1$ and the restriction of $\theta_E$ to $[s_0,s_1]$ are determined as in the proof of Proposition 4.3. Next, the restriction of $\theta_E$ to $[s_1,s_2]$ ($s_2$ as yet unknown) is determined from the initial value problem

$$\begin{aligned} &\dot{\theta}(s) + (-1)^{k_1} [-2A_2 \sin(\theta(s) - \beta_2)]^{\frac{1}{2}} = 0 \\ &\theta(s_1) = \alpha_1 \quad . \end{aligned} \tag{4.14}$$

The solution is the n.r. of a simple elastica with inflection points at equally spaced abscissas $\sigma_k$ where $\theta(\sigma_k) = \beta_2$ or $\beta_2 + \pi \pmod{2\pi}$. By (4.12i and iii) we have

$$\sin(\alpha_1 - \beta_2) < 0, \quad \sin(\alpha_2 - \beta_2) < 0 \quad ,$$

and, therefore, $\theta(s)$ attains the values $\alpha_1$ and $\alpha_2$ exactly once between any two consecutive $\sigma_k$. Thus if there are $k_2 \geq 1$ inflection points between $s_1$ and $s_2$ there is exactly one $s_2$ for which $\theta(s_2) = \alpha_2$. If $k_2 = 0$ and, say $\dot\theta(s_1) > 0$, then by condition (4.12ii) $\sin(\alpha_2 - \alpha_1) > 0$, which implies that $\theta(s)$ attains the value $\alpha_2$ for $s_2 > s_1$, with no inflection point between $s_1$ and $s_2$. By the same arguments the values of $s_3,\ldots,s_{m-1}$ and the restriction of $\theta_E$ to $[s_3,s_4],\ldots,[s_{m-2},s_{m-1}]$ are determined. $s_m = s_E$ and $\theta_E$ on $[s_{m-1},s_m]$ are similarly obtained, except that the condition $\theta(s_m) = \alpha_m$ is replaced by $\dot\theta(s_m) = 0$. The obtained function $\theta_E$ is in $C^1[0,s_E]$ because of condition (4.12iii), and it satisfies (4.13) and conditions A. and C. of Proposition 4.2 by construction.

Proposition 4.5. The projection $M_{|\mathcal{S}}$ is a local diffeomorphism onto an open subset $\mathcal{M}_{(k_1,\ldots,k_m)}$ of $\tilde{\mathbb{R}}_+^m \times \tilde{T}^m$. Thus $\mathcal{S}$ is a 2m-dimensional smooth (even analytic) manifold ($\mathcal{S}$ is not connected if $m \geq 1$). The composition map $M \circ J$ is a (global) homeomorphism of $\mathcal{E}_{(k_1,\ldots,k_m)}$ onto $\mathcal{M}_{(k_1,\ldots,k_m)}$.

Proof: Choose any $(\alpha^0,B^0) \in \mathcal{S}$, $\alpha^0 = (\alpha_1^0,\ldots,\alpha_m^0)$ and $B^0 = (A_1^0,\ldots,A_m^0,\ \beta_1^0,\ldots,\beta_m^0)$. Let $U$ denote the open subset of $T^{m-1} \times (\tilde{\mathbb{R}}_+^m \times \tilde{T}^m)$ satisfying conditions (4.12i,ii). Define a mapping $\varphi : U \to \mathbb{R}^{m-1}$ by

$$\varphi_i(\alpha,B) = A_i \sin(\alpha_i - \beta_i) - A_{i+1} \sin(\alpha_i - \beta_{i+1}), \quad i = 1,\ldots,m-1 \quad . \tag{4.15}$$

With this notation, $(\alpha,B) \in U$ is in $\mathcal{S}$ if and only if $\varphi_i(\alpha,B) = 0$ $(i = 1,\ldots,m-1)$. Now $\varphi_i(\alpha^0,B^0) = 0$ and the Jacobian $[\partial\varphi_i/\partial\alpha_j]$ is nonsingular at $(\alpha^0,B^0)$, since it is a diagonal matrix with diagonal entries

$$\begin{aligned}\frac{\partial\varphi_i}{\partial\alpha_i}(\alpha^0,B^0) &= A_i^0 \cos(\alpha_i^0 - \beta_i^0) - A_{i+1}^0 \cos(\alpha_i^0 - \beta_{i+1}^0) \\ &= \Delta_i \dot\kappa \neq 0 \quad .\end{aligned} \tag{4.16}$$

We conclude there is, for every neighborhood $U_0 \subset U$ of $(\alpha^0,B^0)$ a neighborhood $N_0$ of $B^0$ in $\tilde{\mathbb{R}}_+^m \times \tilde{T}^m$ and a $C^1$-mapping $\alpha$ of $N_0$ such that $\alpha(B^0) = \alpha^0$ and $\varphi_i(\alpha(B),B) = 0$

for all $B \in N_0$. This proves that $M$ is a local diffeomorphism.

By Proposition 4.3 the mapping $J$ is a homeomorphism of $E_{(k_1,\ldots,k_m)}$ onto $S$. If the composite map $M \circ J$ is not a homeomorphism, there must be $(\alpha,B)$, $(\alpha',B)$ in $S$ with $\alpha \neq \alpha'$, say $\alpha_i \neq \alpha_i'$ (mod $2\pi$). By (4.12iii) we have $A_i \sin(\alpha_i - \beta_i) - A_{i+1} \sin(\alpha_i - \beta_{i+1}) = 0$ and $A_i \sin(\alpha_i' - \beta_i) - A_{i+1} \sin(\alpha_i' - \beta_{i+1}) = 0$. These equations imply $\alpha_i' = \alpha_i + \pi$ (mod $2\pi$). But by (4.12i), $\sin(\alpha_i - \beta_i) < 0$ and $\sin(\alpha_i' - \beta_i) < 0$, which contradicts the previous conclusion. Thus Proposition 4.5 is completely proved.

Remark 4.3. It seems to be difficult to give an intrinsic characterization of the set $M_{(k_1,\ldots,k_m)}$. Examples show that it does not coincide with $\tilde{\mathbb{R}}_+^m \times \tilde{T}^m$. It certainly contains points $B = (A,\beta)$ for each combination of the inequalities $A_1 \gtrless A_2, A_2 \gtrless A_3, \ldots, A_{m-1} \gtrless A_m$. Thus, $M_{(k_1,\ldots,k_m)}$ and $S_{(k_1,\ldots,k_m)}$ have at least $2^{m-1}$ disjoint components.

Remark 4.4. The following examples show that the results of this section fail if in Definition 4.1 either (4.1i) or (4.1ii) is omitted.

Example 4.1. Consider the 3-point interpolant $E_0$ with n.r.

$$\dot{\theta}_0(s) - [-2 \sin \theta_0(s)]^{\frac{1}{2}} = 0, \quad 0 \leq s \leq s_1 = \int_0^\pi \frac{du}{\sqrt{2 \sin u}}$$

$$\dot{\theta}_0(s) + [-\sin \theta_0(s)]^{\frac{1}{2}} = 0, \quad s_1 \leq s \leq s_1 + \int_0^\pi \frac{du}{\sqrt{\sin u}} \quad .$$

Here $B_{E_0} = (1,0,\frac{1}{2},0)$, $\alpha_0 = \pi$, $\alpha_1 = 2\pi$, $\alpha_2 = \pi$. The mode is $(0,0)$. $E_0$ violates (4.1i) since $\dot{\theta}_0(s_1) = 0$. For $\varepsilon > 0$ let the extremal $E_\varepsilon$ of the same mode $(0,0)$ be given by $B_{E_\varepsilon} = (1,0,\frac{1}{2},\varepsilon)$, so that

$$\dot{\theta}_\varepsilon(s) - [-2 \sin \theta_\varepsilon(s)]^{\frac{1}{2}} = 0, \quad 0 \leq s \leq s_{1,\varepsilon} \quad ,$$

$$\dot{\theta}_\varepsilon(s) + [-\sin(\theta_\varepsilon(s) - \varepsilon)]^{\frac{1}{2}} = 0, \quad s_{1,\varepsilon} \leq s \leq s_{2,\varepsilon} \quad .$$

If $\theta_\varepsilon(s_{1,\varepsilon})$ is close to $\alpha_1 = 2\pi$ then $\theta_\varepsilon(s_{1,\varepsilon}) = 2\pi - \delta$ for some $\delta > 0$. Thus, for $s = s_{1,\varepsilon}$,

$$\dot{\theta}_\varepsilon(s_{1,\varepsilon}) = [2 \sin \delta]^{\frac{1}{2}} = -[\sin(\delta + \varepsilon)]^{\frac{1}{2}} ,$$

which is impossible. Thus, $\alpha_1(B)$ cannot be defined as a continuous function in a full neighborhood of $B_0$.

Example 4.2. Choose $\pi < \alpha_* < 2\pi$ and consider the 3-point interpolant $E_*$ with n.r.

$$\dot{\theta}_*(s) - [-2 \sin \theta_*(s)]^{\frac{1}{2}} = 0, \quad 0 \le s \le s_{1_*} = \int_0^{\alpha_*} \frac{du}{\sqrt{2 \sin u}}$$

$$\dot{\theta}_*(s) - [-2 \sin \theta_*(s)]^{\frac{1}{2}} = 0, \quad s_{1_*} \le s \le s_2 = 2 \int_0^{\pi} \frac{du}{\sqrt{2 \sin u}} .$$

Here $B_{E_*} = (1,0,1,0)$, $\alpha_0 = \pi$, $\alpha_1 = \alpha_*$, $\alpha_2 = \pi$. The mode is $(0,1)$. $E_*$ violates (4.1ii) since $A_1 = A_2$. For $\pi < \tilde{\alpha} < 2\pi$ let $\tilde{E}$ be defined by the n.r.:

$$\dot{\tilde{\theta}}(s) - [-2 \sin \tilde{\theta}(s)]^{\frac{1}{2}} = 0, \quad 0 \le s \le \tilde{s}_1 = \int_0^{\tilde{\alpha}} \frac{du}{\sqrt{2 \sin u}}$$

$$\dot{\tilde{\theta}}(s) - [-2 \sin \tilde{\theta}(s)]^{\frac{1}{2}} = 0, \quad \tilde{s}_1 \le s \le s_2 = 2 \int_0^{\pi} \frac{du}{\sqrt{2 \sin u}} .$$

Here $B_{\tilde{E}} = (1,0,1,0)$, $\alpha_0 = \pi$, $\alpha_1 = \tilde{\alpha}$, $\alpha_2 = \pi$, and the mode is $(0,1)$ as before. Since there are extremals for all $\pi < \tilde{\alpha} < 2\pi$, $\tilde{E}$ cannot be defined by $B$ and its mode.

## §5. Perturbations of Configurations

In the last section it was seen that the extremal interpolants with $m$ variable interpolation nodes (more precisely, those that belong to a fixed class $E_{(k_1,\ldots,k_m)}$) form a $2m$-dimensional manifold. One expects that an arbitrary configuration $P = \{0,p_1,\ldots,p_m\}$ can be interpolated by an extremal interpolant (possibly by one from each class $E_{(k_1,\ldots,k_m)}$). No solution of any kind exists for this existence problem. In this section we investigate existence in the small. Does the set of configurations $P$ for which extremal interpolants exist have nonempty interior in $\mathbb{R}^{2m}$? More specifically, which $P$ in $\mathbb{R}^{2m}$ are interior points of this set?

To attack this perturbation problem one is tempted to consider the mapping from the $2m$-dimensional set $M_{(k_1,\ldots,k_m)}$ that coordinatizes the elements of $E_{(k_1,\ldots,k_m)}$ (see §4), or from another $2m$-dimensional set of parameters, to the configurations in $\mathbb{R}^{2m}$ which are interpolated. However, this mapping is so complicated - it involves the elliptic integrals which are the solutions of the extremal equations - that little insight is gained from its consideration. For this reason we start with the extremals themselves, as defined by their differential equations.

Let $\bar{\theta} : [0,\bar{s}_m] \to T^1$ be the normal representation of a given extremal interpolant $\bar{E}$, which interpolates the configuration

$$\bar{P} = \{0,\bar{p}_1,\ldots,\bar{p}_m\} \quad ,$$

so that

$$\int_0^{\bar{s}_i} \cos\bar{\theta} = \bar{p}_i^1, \quad \int_0^{\bar{s}_i} \sin\bar{\theta} = \bar{p}_i^2, \quad i = 0,1,\ldots,m \tag{5.1}$$

$$0 = \bar{s}_0 < \bar{s}_1 < \cdots < \bar{s}_m \quad .$$

Since we consider only extremals $E$ with n.r. $\theta$ near $\bar{\theta}$, hence with knots $s_i$ near $\bar{s}_i$, we choose $\bar{\varepsilon} > 0$, $\bar{\varepsilon} = \frac{1}{2}\min(\bar{s}_i - \bar{s}_{i-1})$ and extend $\bar{\theta}$ to the interval $[0,\bar{s}]$, $\bar{s} = \bar{s}_m + \bar{\varepsilon}$, by setting

$$\bar{\theta}(s) = \bar{\theta}(\bar{s}_m), \quad \bar{s}_m < s \le \bar{s} \quad .$$

We introduce two spaces of mappings from the interval $[0,\bar{s}]$:

NBV = space of functions $\kappa : [0,\bar{s}] \to \mathbb{R}$ of bounded variation $V(\kappa)$ and continuous from the right with $\kappa(0) = \kappa(\bar{s} - 0) = 0$ and norm $V(\kappa)$.

$NBV_1$ = space of functions $\theta : [0,\bar{s}] \to T^1$, which are locally absolutely continuous and have derivatives $\dot{\theta} \in NBV$, with norm $\sup|\theta| + V(\dot{\theta})$.

Both NBV and $NBV_1$ are B-spaces. Clearly $\bar{\theta}$ as defined above is in $NBV_1$ and $\dot{\bar{\theta}}$ is in NBV.

If $\theta \in NBV_1$ is the normal representation of an extremal E which interpolates the configuration $P = \{0,p_1,\ldots,p_m\}$ at the nodes $0 = s_0 < s_1 < \cdots < s_m < \bar{s}$ (more precisely, we speak of the linear extension of E to length $\bar{s}$) then the following equations hold (see Proposition 2.4):

$$\text{(i)} \quad \begin{aligned} &\dddot{\theta}(s) + \tfrac{1}{2}\dot{\theta}^3(s) = 0, && \text{for } s_{i-1} < s < s_i, \ i = 1,\ldots,m \\ &\dddot{\theta}(s) = 0, && \text{for } s_m < s < \bar{s} \end{aligned}$$

$$(5.2) \qquad \text{(ii)} \quad \dot{\theta}(s_i - 0) - \dot{\theta}(s_i) = 0, \quad i = 1,\ldots,m .$$

$$\text{(iii)} \quad \int_0^{s_i} \cos\theta = p_i^1, \quad \int_0^{s_i} \sin\theta = p_i^2, \quad i = 1,\ldots,m .$$

It is easy to show that these equations characterize the interpolant E completely.

We rewrite equations (5.2) by using the values

$$(5.3) \qquad \dot{\theta}(s_i) = a_i, \quad \ddot{\theta}(s_i + 0) = b_i, \quad i = 1,\ldots,m$$

as parameters (but $a_0 = 0$, $b_m = 0$ always).

$$(5.4)\quad \begin{aligned}
&\text{(i)} \quad \begin{cases} \dot\theta(s) + \frac{1}{2}\int_{s_{i-1}}^{s} (s-t)\dot\theta^3(t)dt = a_{i-1} + b_{i-1}(s-s_{i-1}), & s_{i-1} \le s < s_i, \quad i = 1,\dots,m \\ \dot\theta(s) = 0, & s_m \le s \le \bar{s} \ . \end{cases} \\
&\text{(ii)} \quad a_i = a_{i-1} + b_{i-1}(s_i - s_{i-1}) - \frac{1}{2}\int_{s_{i-1}}^{s_i} (s_i - t)\dot\theta^3(t)dt, \quad i = 1,\dots,m \ . \\
&\text{(iii)} \quad \int_0^{s_i} \cos\theta = p_i^1, \quad \int_0^{s_i} \sin\theta = p_i^2, \quad i = 1,\dots,m \ .
\end{aligned}$$

Equations (5.4) define implicitly a mapping $\mathcal{G}$ from the space $\mathcal{P} \subset (\mathbb{R}^2)^m$ of configurations P to the space $\mathcal{E}$ of extremal interpolants E. To apply the Implicit Function Theorem we introduce a mapping G on the produce space $\mathcal{E} \times \mathcal{P}$ to $NBV \times \mathbb{R}^m \times \mathbb{R}^{2m}$ as follows. We set

$$\Theta = (\theta;\ s_1,\dots,s_m;\ a_1,\dots,a_m;\ b_0,\dots,b_{m-1}), \quad \text{where}$$

$$\theta \in NBV_1;\ s_i \in \mathbb{R},\ a_i \in \mathbb{R},\ b_i \in \mathbb{R}$$

$$D = NBV_1 \times \Pi(\bar{s}_i - \bar\varepsilon,\ \bar{s}_i + \bar\varepsilon) \times \mathbb{R}^m \times \mathbb{R}^m$$

and define a mapping $G = (g, r_i, q_i)$ with components $g \in NBV$, $r_i \in \mathbb{R}$, $q_i \in \mathbb{R}^2$, as follows:

$$(5.5)\quad \begin{aligned}
&\text{(i)} \quad \begin{cases} g(s) = \dot\theta(s) + \frac{1}{2}\int_{s_{i-1}}^{s} (s-t)\dot\theta^3(t)dt - a_{i-1} - b_{i-1}(s-s_{i-1}), & s_{i-1} \le s < s_i \\ g(s) = \dot\theta(s), & s_m \le s \le \bar{s} \ . \end{cases} \\
&\text{(ii)} \quad r_i = a_i - a_{i-1} - b_{i-1}(s_i - s_{i-1}) + \frac{1}{2}\int_{s_{i-1}}^{s_i} (s_i - t)\dot\theta^3(t)dt \ . \\
&\text{(iii)} \quad q_i^1 = \int_0^{s_i} \cos\theta - p_i^1, \quad q_i^2 = \int_0^{s_i} \sin\theta - p_i^2 \ .
\end{aligned}$$

Clearly, Equations (5.4) are equivalent to

$$G(\Theta, P) = 0 \ .$$

In particular we have $G(\bar{\Theta},\bar{P}) = 0$, since we assume that $\bar{\theta}$ is the n.r. of the extremal interpolant $\bar{E}$ for the configuration $\bar{P}$. We need the Fréchet differential $G'_{\Theta}(\Theta,P)[\Psi]$ where

$$\Psi = (\psi;\ t_1,\ldots,t_m;\ \alpha_1,\ldots,\alpha_m;\ \beta_0,\ldots,\beta_{m-1})$$

is an increment to $\Theta$. The components of $G'_{\Theta}(\Theta,P)[\Psi]$ are denoted by $g'$, $r'_i$, $q'_i$. One finds readily

$$\text{(5.6)}\quad \begin{aligned} &\text{(i)} && g'(s) = \dot{\psi}(s) + \frac{3}{2}\int_{s_{i-1}}^{s} (s-t)\kappa^2(t)\dot{\psi}(t)\,dt - \frac{1}{2}\kappa^3_{i-1}(s-s_{i-1})t_{i-1} \\ & && \qquad - \alpha_{i-1} - \beta_{i-1}(s-s_{i-1}) + b_{i-1}t_{i-1}, \quad s_{i-1} \le s < s_i \\ & && g'(s) = \dot{\psi}(s), \quad s_m \le s < \bar{s} \ . \\ &\text{(ii)} && r'_i = \alpha_i - \alpha_{i-1} - \beta_{i-1}\Delta_i s + \left(b_{i-1} - \frac{1}{2}\Delta_i s\kappa^3_{i-1}\right)t_{i-1} - \dot{\kappa}(s_i-0)t_i + \frac{3}{2}\int_{s_{i-1}}^{s_i} (s_i-t)\kappa^2\dot{\psi} \ . \\ &\text{(iii)} && (q^1_i)' = -\int_0^{s_i} \psi \sin\theta + t_i\cos\theta_i, \quad (q^2_i)' = \int_0^{s_i} \psi\cos\theta + t_i \sin\theta_i \ . \end{aligned}$$

Here we have used the notations

$$\text{(5.7)}\qquad \kappa = \dot{\theta},\ \theta_i = \theta(s_i),\ \kappa_i = \dot{\theta}(s_i),\ \Delta_i s = s_i - s_{i-1} \ ,$$

and the relation

$$\frac{1}{2}\int_{s_{i-1}}^{s_i} \dot{\theta}^3 = \ddot{\theta}(s_{i-1}+0) - \ddot{\theta}(s_i - 0) = b_{i-1} - \dot{\kappa}(s_i - 0) \ ,$$

which follows from (5.2i) and (5.3).

The continuity of $G'$ near $(\bar{\Theta},\bar{P})$ is readily ascertained from (5.6).

We can now state the main result of this section.

<u>Theorem 5.1</u>. $G'_{\Theta}(\bar{\Theta},\bar{P})$ is an isomorphism of

$$NBV_1 \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \quad \text{onto} \quad NBV \times \mathbb{R}^m \times (\mathbb{R}^2)^m$$

if and only if $(\bar{\Theta},\bar{P})$ satisfies the following hypothesis:

(A) The system for the unknown $\psi \in NBV_1$:

$$(5.8)\qquad \begin{aligned} &\text{(i)} \quad \begin{cases} \dddot{\psi} + \frac{3}{2}\bar{\kappa}^2\dot{\psi} = 0 & \text{on } (\bar{s}_{i-1},\bar{s}_i),\quad i = 1,\ldots,m \\ \dot{\psi} = 0 & \text{on } (\bar{s}_m,\bar{s})\ , \end{cases} \\ &\text{(ii)} \quad \Delta_i\dot{\psi} + \Delta_i\dot{\bar{\kappa}} \int_0^{\bar{s}_i} \psi \sin(\bar{\theta}-\bar{\theta}_i) = 0,\quad i = 1,\ldots,m\ , \\ &\text{(iii)} \quad \int_0^{\bar{s}_i} \psi \cos(\bar{\theta}-\bar{\theta}_i) = 0,\quad i = 1,\ldots,m\ , \end{aligned}$$

has only the trivial solution $\psi = 0$. If (A) is satisfied then there are neighborhoods $N_{\bar{\Theta}}$ in D and $N_{\bar{P}}$ in $(\mathbb{R}^2)^m$ of $\bar{\Theta}$ and $\bar{P}$ respectively and a diffeomorphism $\Pi$,

$$\Pi : N_{\bar{\Theta}} \xrightarrow{\text{onto}} N_{\bar{P}}$$

such that $\Theta \in N_{\Theta}$ defines an extremal P-interpolant E for $P = \Pi(\Theta)$.

<u>Remark 5.1</u>. Explanation of the notations used above and in the following:

$$\bar{\theta}_i = \bar{\theta}(\bar{s}_i),\quad \bar{\kappa}_i = \dot{\bar{\theta}}(\bar{s}_i),\quad \psi_i = \psi(\bar{s}_i)\ ,$$

$$\Delta_i\dot{\bar{\kappa}} = \dot{\bar{\kappa}}(\bar{s}_i+0) - \dot{\bar{\kappa}}(\bar{s}_i-0),\quad \Delta_i\dot{\psi} = \dot{\psi}(\bar{s}_i+0) - \dot{\psi}(\bar{s}_i-0)\ ,$$

in particular $\Delta_0\dot{\bar{\kappa}} = \dot{\bar{\kappa}}(\bar{s}_0+0)$, $\Delta_m\dot{\bar{\kappa}} = -\dot{\bar{\kappa}}(\bar{s}_m-0)$, etc.

<u>Proof</u>: We first demonstrate the injectivity of the bounded linear mapping $G'_{\Theta}(\bar{\Theta},\bar{P})$ under hypothesis (A). Thus, assume that for some $\Psi$:

$$G'_{\Theta}(\bar{\Theta},\bar{P})[\Psi] = 0\ .$$

Then we have by (5.6i):

(5.9i)
$$\dddot{\psi} + \frac{3}{2}\kappa^{-2}\dot{\psi} = 0 \quad \text{on} \quad (\bar{s}_{i-1},\bar{s}_i), \quad i = 1,\dots,m$$
$$\dot{\psi} = 0 \quad \text{on} \quad (\bar{s}_m,\bar{s}) \quad ,$$

(5.10)
$$\dot{\psi}(\bar{s}_i-0) + \frac{3}{2}\int_{\bar{s}_{i-1}}^{\bar{s}_i} (\bar{s}_i-t)\bar{\kappa}^{-2}\dot{\psi} - \frac{1}{2}\bar{\kappa}_{i-1}^{-3}\Delta_i\bar{s}t_{i-1}$$
$$- \alpha_{i-1} - \beta_{i-1}\Delta_i\bar{s} + b_{i-1}t_{i-1} = 0 \quad ,$$

and using (5.6ii)

(5.11i)
$$\dot{\psi}(\bar{s}_i-0) - \alpha_i + \dot{\bar{\kappa}}(\bar{s}_i-0)t_i = 0, \quad i = 1,\dots,m \quad .$$

Also by (5.6i)

(5.11ii)
$$\dot{\psi}(\bar{s}_i+0) - \alpha_i + \bar{b}_i t_i = 0 \quad .$$

Since $\bar{b}_i = \dot{\bar{\kappa}}(\bar{s}_i+0)$, the last two equations yield

(5.12)
$$\Delta_i\dot{\psi} + t_i\Delta_i\dot{\bar{\kappa}} = 0, \quad i = 1,\dots,m \quad .$$

The remaining equations $(q_i^1)' = (q_i^2)' = 0$ (see 5.6iii)) are equivalent to

(5.13)
$$t_i = \int_0^{\bar{s}_i} \psi \sin(\bar{\theta}-\bar{\theta}_i), \quad i = 1,\dots,m \quad .$$

(5.9iii)
$$\int_0^{\bar{s}_i} \psi \cos(\bar{\theta}-\bar{\theta}_i) = 0, \quad i = 1,\dots,m \quad .$$

When (5.13) is substituted in (5.12), one obtains

(5.9ii)
$$\Delta_i\dot{\psi} + \Delta_i\dot{\bar{\kappa}} \int_0^{\bar{s}_i} \psi \sin(\bar{\theta}-\bar{\theta}_i) = 0, \quad i = 1,\dots,m \quad .$$

By hypothesis (A) the equations (5.9i,ii,iii), which coincide with Equations (5.8), together with $\psi \in NBV_1$ imply $\psi = 0$ on $[0,\bar{s}_m]$. (5.9i) then implies $\psi = 0$ also on $[\bar{s}_m,\bar{s}]$. Then by (5.13), $t_i = 0$ $(i = 1,\dots,m)$, and by (5.11), $\alpha_i = 0$ $(i = 1,\dots,m)$. Finally, by (5.10),

$\beta_i = 0 \ (i = 0,\dots,m-1)$, thus $\Psi = 0$, hence $G'_{\Theta}(\bar{\Theta},\bar{P})$ is injective.

Conversely if hypothesis (A) is not satisfied, i.e. system (5.9i,ii,iii) has a nontrivial solution $\psi \in NBV_1$, then determine the $t_i$ from (5.13), the $\alpha_i$ from (5.11) and the $\beta_i$ from (5.10). $\Psi = (\psi;\ t_1,\dots,t_m;\ \alpha_1,\dots,\alpha_m;\ \beta_0,\dots,\beta_{m-1})$ is then a nontrivial solution of $G'_{\Theta}(\bar{\Theta},\bar{P})[\Psi] = 0$, thus $G'_{\Theta}(\bar{\Theta},\bar{P})$ is not an isomorphism.

To show surjectivity of $G'_{\Theta}(\bar{\Theta},\bar{P})$ onto $NBV \times \mathbb{R}^m \times \mathbb{R}^{2m}$, assume we are given $h \in NBV$, $u_i \in \mathbb{R}$, $v_i \in \mathbb{R}^2$ $(i = 1,\dots,m)$. We must find $\psi \in NBV_1$, $t_i \in \mathbb{R}$, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$, such that (see 5.6)

$$\text{(i)} \quad g'(s) = h(s), \quad 0 \le s < \bar{s} \quad .$$

$$\text{(5.14)} \qquad \text{(ii)} \quad r_i' = u_i, \quad i = 1,\dots,m \quad .$$

$$\text{(iii)} \quad (q_i^1)' = v_i^1, \ (q_i^2)' = v_i^2, \quad i = 1,\dots,m \quad .$$

In particular, (5.14i) requires (see 5.6i)

$$\text{(5.15i)} \qquad \begin{aligned} &\dot{\psi}(\bar{s}_i-0) + \frac{3}{2}\int_{\bar{s}_{i-1}}^{\bar{s}_i} (\bar{s}_i-t)\bar{\kappa}^2\dot{\psi} - \frac{1}{2}\bar{\kappa}_{i-1}^3\Delta_i\bar{s}t_{i-1} \\ &\quad - \alpha_{i-1} - \beta_{i-1}\Delta_i\bar{s} + \bar{b}_{i-1}t_{i-1} = h(\bar{s}_i-0) \end{aligned}$$

and (5.14ii) rquires (see 5.6ii)

$$\begin{aligned} &\alpha_i - \alpha_{i-1} - \beta_{i-1}\Delta_i\bar{s} + (\bar{b}_{i-1} - \frac{1}{2}\Delta_i\bar{s}\bar{\kappa}_{i-1}^3)t_{i-1} \\ &\quad - \dot{\bar{\kappa}}(\bar{s}_i-0)t_i + \frac{3}{2}\int_{\bar{s}_{i-1}}^{\bar{s}_i} (\bar{s}-t)\bar{\kappa}^2\dot{\psi} = u_i \quad . \end{aligned}$$

The last two equations imply

$$\text{(5.16i)} \qquad \dot{\psi}(\bar{s}_i-0) - \alpha_i + \dot{\bar{\kappa}}(\bar{s}_i-0)t_i = h(\bar{s}_i-0) - u_i \quad .$$

(5.14i) also requires

(5.16ii) $$\dot\psi(\bar s_i+0) - \alpha_i + \hat\kappa(\bar s_i+0)t_i = h(\bar s_i+0) \quad .$$

We therefore must have

(5.17i) $$\dot\psi(\bar s_i+0) = \dot\psi(\bar s_i-0) - \Delta_i\hat\kappa t_i + \Delta_i h + u_i, \quad i = 1,\ldots,m \quad .$$

The last equations (5.14iii) are (see 5.6iii)

(5.17ii) $$-\int_0^{\bar s_i} \psi \sin \bar\theta + t_i \cos \bar\theta_i = v_i^1, \int_0^{\bar s_i} \psi \cos \bar\theta + t_i \sin \bar\theta_i = v_i^2 \quad .$$

The general solution $\psi \in NBV_1$ of (5.14i) is the sum of a particular solution and the linear combination of 3m functions with the coefficients $\alpha_1,\ldots,\alpha_m, \beta_0,\ldots,\beta_{m-1}, t_1,\ldots,t_m$. When this $\psi$ is substituted in (5.17i and ii) a nonhomogeneous system of 3m equations for the unknowns $\alpha_i, \beta_i, t_i$ is obtained. The homogeneous part of this system corresponds to the case $h = 0$, $u_i = 0$, $v_i = 0$, and it has only the trivial solution $\alpha_i = 0$, $\beta_i = 0$, $t_i = 0$, as shown in the first part of the proof. This demonstrates the surjectivity of $G_\Theta'(\bar\Theta,\bar P)$ and finishes the proof of Theorem 5.1.

The utility of this theorem is illustrated by the fact that it readily implies the following important result.

<u>Corollary 5.2</u>. Suppose $\bar P = (0, \bar p_1,\ldots,\bar p_m)$ is the ray configuration $\bar p_i = (\bar s_i,0)$ $(i = 1,\ldots,m)$ with the trivial interpolant $\bar\Xi$. Then hypothesis (A) is satisfied, hence the conclusion of Theorem 5.1 holds.

<u>Proof</u>: $\bar\theta = 0$ in this case. If we put

(5.18) $$s = x,\ \bar s_i = x_i,\ \bar s = \bar x,\ y(x) = \int_0^x \psi$$

then $y(0) = 0$ and Equations (5.8) become

(5.19) $$\begin{aligned} &y^{(4)} = 0 \quad \text{on} \quad (x_{i-1},x_i), \quad i = 1,\ldots,m \\ &y'' = 0 \quad \text{on} \quad (x_m,\bar x) \\ &y''(x_i+0) - y''(x_i-0) = 0, \quad i = 1,\ldots,m \\ &y(x_i) = 0 \qquad\qquad i = 1,\ldots,m \end{aligned}$$

while $y' \in NBV_1$, i.e. $y'' \in NBV$, in particular $y''(0) = 0$. These are exactly the equations for a natural cubic spline that interpolates the points $(x_i,0)$ $(i = 0,\ldots,m)$. It follows that $y(x) \equiv 0$, and the corollary is proved.

Remark 5.2. We briefly draw a connection between the above corollary and natural cubic spline interpolation. The mapping $\Gamma$ from the space of configurations $P$ to the space of extremal interpolants $E$ is implicitly defined by $G(\Theta,P) = 0$. Perturbation theory looks for a pair $\Theta = \bar{\Theta} + \Psi$, $P = \bar{P} + Z$ $(Z = \{z_1,\ldots,z_m\})$ close to the initial pair $\bar{\Theta},\bar{P}$ for which

$$(5.20) \qquad G'_{\Theta}(\bar{\Theta},\bar{P})[\Psi] + G'_P(\bar{\Theta},\bar{P})[Z] = 0 \quad .$$

It is readily seen that this is system (5.9i,ii,iii) except that (5.9ii) is replaced by

$$(5.21i) \qquad \int_0^{\bar{s}_i} \psi \cos(\bar{\theta}-\bar{\theta}_i) = z_i^2 \cos \bar{\theta}_i - z_i^1 \sin \bar{\theta}_i \quad .$$

Also, Equation (5.13) is replaced by

$$(5.21ii) \qquad t_i = \int_0^{\bar{s}_i} \psi \sin(\bar{\theta}-\bar{\theta}_i) + z_i^1 \cos \bar{\theta}_i + z_i^2 \sin \bar{\theta}_i \quad .$$

Now suppose the initial configuration $\bar{P}$ and the interpolant $\bar{E}$ are the ray configuration and trivial interpolant, as in Corollary 5.2. Further suppose $P$ is the configuration

$$P = \{(x_0,y_0),\ (x_1,y_1),\ldots,(x_m,y_m)\},\ x_0 = y_0 = 0$$

with the $x_i - \bar{s}_i = z_i^1$ and $y_i = z_i^2$ small. Then (5.21i,ii) become (in the perturbation approximation)

$$(5.22) \qquad \int_0^{x_i} \psi \approx y_i, \quad \bar{s}_i + t_i \approx x_i \quad ,$$

moreover

$$(5.23) \qquad x(s) = \int_0^s \cos\theta = \int_0^s \cos\psi \approx s,\ y(s) = \int_0^s \sin\theta = \int_0^s \sin\psi \approx \int_0^s \psi \quad .$$

The perturbed extremal $E$ is now the graph of the function $x \to y(x)$, which satisfies Equations (5.19), except that the last equation is replaced by $y(x_i) = y_i$. Thus $y$ is the natural cubic spline that interpolates the data $(x_i, y_i)$ $(i = 0, 1, \ldots, m)$.

We have shown that the cubic spline interpolant is the result of linerization of extremal interpolation (in the sense of making $\int \kappa^2 ds$ stationary) near the trivial interpolant for the ray configuration.

<u>Remark 5.3</u>. The differential equation that appears in hypothesis (A)

$$\dddot{\psi} + \frac{3}{2}\kappa^2\dot{\psi} = 0 \tag{5.24i}$$

where $\kappa = \dot{\theta}$ is defined by the equation

$$\dddot{\theta} + \frac{1}{2}\dot{\theta}^3 = 0 \tag{5.24ii}$$

can be completely integrated by quadratures alone. Indeed $\psi = 1$ is clearly one integral;

$$\psi_1 = \kappa \tag{5.25i}$$

is another; for $\dddot{\psi}_1 = \theta^{(4)}$ and $\theta^{(4)} + (3/2)\dot{\theta}^2\ddot{\theta} = 0$. The third one is

$$\psi_2(s) = s\kappa(s) \quad . \tag{5.25ii}$$

In the special case where $\kappa(s) \equiv cs$, three linearly independent integrals are

$$1,\ s,\ s^2 \quad . \tag{5.25iii}$$

In the remainder of this section we discuss the replacement of hypothesis (A) by a simpler condition, which requires only that the value of an explicitly given function (involving many quadratures) at $\bar{s}_m$ be $\neq 0$. For this purpose we introduce a condition on arcs of simple elastica. The arc of $\bar{E}$ from $\bar{s}_{i-1}$ to $\bar{s}_i$ is said to be <u>ordinary</u> if it is either straight or

$$\begin{aligned}\delta_i := {} & (\bar{p}_i^1 - \bar{p}_{i-1}^1)[(\bar{\kappa}_{i-1})^2\cos\bar{\theta}_i + \hat{\kappa}_{i-1}\sin\bar{\theta}_i] \\ & + (\bar{p}_i^2 - \bar{p}_{i-1}^2)[(\bar{\kappa}_{i-1})^2\sin\bar{\theta}_i - \hat{\kappa}_{i-1}\cos\bar{\theta}_i] + \hat{\kappa}_{i-1}\sin(\bar{\theta}_{i-1} - \bar{\theta}_i) \neq 0 \quad .\end{aligned} \tag{5.26}$$

One sees readily that only exceptionally is such an arc not ordinary. For example, the arc from $\bar{s}_0 = 0$ to $\bar{s}_1$, where $\bar{\kappa}_0 = 0$, $\dot{\bar{\kappa}}_0 \neq 0$, is not ordinary if and only if

$$\bar{p}_1^{-1} \sin \bar{\theta}_1 - \bar{p}_1^{-2} \cos \bar{\theta}_1 = 0 \quad ,$$

i.e., the chord $\overline{p_0 p_1}$ is tangent to $E$ at $\bar{p}_1$.

Theorem 5.3. Suppose the extremal $\bar{E}$ consists of ordinary arcs only. Let $\psi_0, \psi_0(0) = 1$, be the solution of system (5.8), exclusive of the condition on $\Delta_m \dot{\psi}$ ($\psi_0$ is explicitly constructed below). Then hypothesis (A) of Theorem 5.1 is satisfied if and only if

$$\dot{\psi}_0(\bar{s}_m - 0) + \dot{\bar{\kappa}}(\bar{s}_m - 0) \int_0^{\bar{s}_m} \psi_0 \sin(\bar{\theta} - \bar{\theta}_m) \neq 0 \quad . \tag{5.27}$$

Proof: Let $\phi_i$, $\chi_i$ be the integrals of Equation $\dddot{\psi} + (3/2)\bar{\kappa}^{-2}\dot{\psi} = 0$ (case $\kappa \neq 0$) for which

$$(5.28) \qquad \begin{aligned} \phi_i(\bar{s}_{i-1}) &= \dot{\phi}_i(\bar{s}_{i-1} + 0) = 0, \quad \ddot{\phi}_i(\bar{s}_{i-1} + 0) = 1 \\ \chi_i(\bar{s}_{i-1}) &= \ddot{\chi}_i(\bar{s}_{i-1} + 0) = 0, \quad \dot{\chi}_i(\bar{s}_{i-1} + 0) = 1 \quad . \end{aligned}$$

These are line... combinations of $1, \psi_1, \psi_2$. One finds easily

$$(5.29\text{i}) \qquad \begin{aligned} \phi_i &= \rho_i[(\bar{\kappa}_{i-1})^2 - \bar{\kappa}_{i-1}\bar{\kappa} + \dot{\bar{\kappa}}_{i-1}(s - \bar{s}_{i-1})\bar{\kappa}] \\ \chi_i &= \rho_i[-2\bar{\kappa}_{i-1}\dot{\bar{\kappa}}_{i-1} + 2\dot{\bar{\kappa}}_{i-1}\bar{\kappa} + \tfrac{1}{2}(\bar{\kappa}_{i-1})^3(s - \bar{s}_{i-1})\bar{\kappa}] \end{aligned}$$

where we have used the abbreviation

$$1/\rho_i = 2(\dot{\bar{\kappa}}_{i-1})^2 + \tfrac{1}{2}(\bar{\kappa}_{i-1})^4 \quad . \tag{5.29ii}$$

We construct $\psi_0$ successively on the intervals $[\bar{s}_{i-1}, \bar{s}_i]$ $(i = 1, \ldots, m)$. On $[0, \bar{s}_1]$ we have since $\psi_0(0) = 1$, $\dot{\psi}_0(0) = 0$:

$$\psi_0 = 1 + \ddot{\psi}_0(0)\phi_1 \tag{5.30i}$$

with $\psi_0$ satisfying condition (5.9iii):

$$(5.30\text{ii})\qquad \int_0^{\bar{s}_1} \cos(\bar{\theta}-\bar{\theta}_0) + \ddot{\psi}_0(0) \int_0^{\bar{s}_1} \phi_1 \cos(\bar{\theta}-\bar{\theta}_0) = 0 \quad .$$

One finds the last integral to be $\rho_1\delta_1$. Thus, $\ddot{\psi}_0(0)$ is uniquely determined from (5.30ii), and (5.30i) gives $\psi_0$ on $(0,\bar{s}_1)$. (If $\bar{\kappa} = 0$ on $[0,\bar{s}_1]$ then one finds $\psi_0(s) = 1 - 3(s/\bar{s}_1)^2$). Assume $\psi_0$ has been determined on $[0,\bar{s}_{i-1}]$ $(i < m)$. Then $\psi_0(\bar{s}_{i-1})$ and $\dot{\psi}_0(\bar{s}_{i-1} - 0)$ are known, hence $\dot{\psi}_0(\bar{s}_{i-1})$ can be found from condition (5.8ii).

$$(5.31)\qquad \dot{\psi}_0(\bar{s}_{i-1}) = \dot{\psi}_0(\bar{s}_{i-1}-0) + \Delta_{i-1}\dot{\bar{\kappa}} \int_0^{\bar{s}_{i-1}} \psi_0 \sin(\bar{\theta}-\bar{\theta}_{i-1}) \quad .$$

Then if $\kappa \neq 0$ on $[\bar{s}_{i-1},\bar{s}_i]$ we have

$$(5.32\text{i})\qquad \psi_0 = \psi_0(\bar{s}_{i-1}) + \dot{\psi}_0(\bar{s}_{i-1})\chi_i + \ddot{\psi}_0(\bar{s}_{i-1})\phi_i \quad ,$$

and to satisfy condition (5.9iii):

$$(5.32\text{ii})\qquad \begin{aligned} &\int_0^{\bar{s}_{i-1}} \psi_0 \cos(\bar{\theta}-\bar{\theta}_i) + \int_{s_{i-1}}^{\bar{s}_i} [\psi_0(\bar{s}_{i-1}) + \dot{\psi}_0(\bar{s}_{i-1})\chi_i]\cos(\bar{\theta}-\bar{\theta}_i) \\ &\quad + \ddot{\psi}_0(\bar{s}_{i-1})\int_{s_{i-1}}^{\bar{s}_i} \phi_i \cos(\bar{\theta}-\bar{\theta}_i) = 0 \quad . \end{aligned}$$

One finds, using (5.29i) and

$$\int_{s_{i-1}}^{\bar{s}_i} \cos\bar{\theta} = \bar{p}_i^1 - \bar{p}_{i-1}^1, \quad \int_{s_{i-1}}^{\bar{s}_i} \sin\bar{\theta} = \bar{p}_i^2 - \bar{p}_{i-1}^2$$

that the last integral in (5.32ii) is $\rho_i\delta_i \neq 0$. Thus $\ddot{\psi}_0(\bar{s}_{i-1})$ is uniquely determined from (5.32ii), and (5.32i) gives $\psi_0$ on $[\bar{s}_{i-1},\bar{s}_i]$.

In the omitted case $\bar{\kappa} = 0$ on $[\bar{s}_{i-1},\bar{s}_i]$, (5.32iii) are replaced by

$$(5.33)\quad \begin{aligned} &\text{(i)}\quad \psi_0(s) = \psi_0(\bar{s}_{i-1}) + \dot{\psi}_0(\bar{s}_{i-1})(s-\bar{s}_{i-1}) + \tfrac{1}{2}\ddot{\psi}_0(\bar{s}_{i-1})(s-\bar{s}_{i-1})^2 \\ &\text{(ii)}\quad \int_0^{\bar{s}_{i-1}} \psi_0 \cos(\bar{\theta}-\bar{\theta}_i) + \psi_0(\bar{s}_{i-1})(\bar{s}_i-\bar{s}_{i-1}) + \tfrac{1}{2}\dot{\psi}_0(\bar{s}_{i-1})(\bar{s}_i-\bar{s}_{i-1})^2 \\ &\qquad + \tfrac{1}{6}\ddot{\psi}_0(\bar{s}_{i-1})(\bar{s}_i-\bar{s}_{i-1})^3 = 0 \ . \end{aligned}$$

The conclusions remain the same as before.

With $\psi_0$ found on $[0,\bar{s}_m]$ there remains condition (5.9ii) to be satisfied:

$$(5.34)\qquad \dot{\psi}_0(\bar{s}_m-0) + \hat{\kappa}(\bar{s}_m-0)\int_0^{\bar{s}_m} \psi_0 \sin(\bar{\theta}-\bar{\theta}_m) = 0 \ .$$

System (5.6) has a nontrivial solution if and only if the constructed integral $\psi_0$ satisfies (5.34). This proves the theorem.

<u>Example 5.1</u>. To illustrate the utility of the preceding theorem consider the configuration

$$P = \{(0,0),\ (a,0),\ (a,b)\}$$

with the extremal interpolant $\bar{E}$ whose n.r. $\bar{\theta}$ is defined by:

$$(5.35)\quad \begin{aligned} &\bar{\theta}(s) = 0 && 0 \le s \le a \ , \\ &\dot{\bar{\theta}}(s) = (\beta/b)[\sin\bar{\theta}(s)]^{1/2}, && a \le s \le \bar{s} \ . \end{aligned}$$

Here $\beta$ and $\bar{s}$ are the definite integrals.

$$\beta = \int_0^\pi \sin^{1/2},\ \bar{s} = a + (b/\beta)\int_0^\pi \sin^{-1/2} \ .$$

That $\bar{E}$ does indeed interpolate the point $(a,b)$ follows from

$$\int_a^{\bar{s}} \cos\bar{\theta}\, ds = \int_0^\pi (1/\dot{\bar{\theta}})\cos t\, dt = (b/\beta)\int_0^\pi \sin^{-1/2}\cos = 0 \ ,$$

$$\int_a^{\bar{s}} \sin\bar{\theta}\, ds = (b/\beta)\int_0^\pi \sin\cdot\sin^{-1/2} = b \ .$$

Using the construction of the preceding theorem, one finds by straight forward computation:

$$\psi_0(s) = 1 - 3(s/a)^2, \quad 0 \le s \le a \quad ,$$

$$\phi_2(s) = (b/\beta)^2(s-a)\dot{\bar{\theta}}(s), \quad X_2(s) = 2(b/\beta)^2\dot{\bar{\theta}}(s) \quad ,$$

$$\psi_0(s) = -2 - (12/a)(b/\beta)^2\dot{\bar{\theta}}(s), \quad a \le s \le \bar{s} \quad ,$$

$$\dot{\psi}_0(\bar{s}-0) + \dot{\bar{\kappa}}(\bar{s}-0)\int_0^{\bar{s}}\psi_0 \sin(\bar{\theta}-\bar{\theta}_2) = \frac{6}{ab} + \frac{\beta^2}{b} > 0 \quad .$$

By Theorem 5.2 unrestricted perturbation of the 3-point rectangular configuration $\bar{P}$ is possible.

Even for this simple example we were not able to prove this result in the more direct way, by expressing the parameters of the elastica spline in terms of the coordinates of the interpolated configuration.

Example 5.2. Let $\tilde{P}$ be the "hair pin configuration"

$$\tilde{P} = \{(0,0),\ (0,\delta),\ (0,0)\}, \quad \delta = \int_0^\pi (2\sin)^{1/2} \quad ,$$

with the extremal interpolant $\tilde{E}$ whose n.r. $\tilde{\theta}$ is defined by:

$$\tilde{\theta}(0) = 0$$

$$\dot{\tilde{\theta}}(s) = [2\sin\tilde{\theta}(s)]^{1/2}, \quad 0 \le s \le \sigma := \int_0^\pi (2\sin)^{-1/2}$$

$$= [-2\sin\tilde{\theta}(s)]^{1/2}, \quad \sigma \le s \le 2\sigma \quad .$$

Then $\tilde{\kappa}(0) = \tilde{\kappa}(\sigma) = \tilde{\kappa}(2\sigma) = 0$, $\Delta\dot{\tilde{\kappa}}_1 = \dot{\tilde{\kappa}}(\sigma+0) - \dot{\tilde{\kappa}}(\sigma-0) = 2$. One finds readily:

$$\phi_1(s) = \frac{s}{2}\tilde{\kappa}(s) \quad , \quad X_1(s) = \tilde{\kappa}(s), \quad 0 \le s \le \sigma \quad ,$$

$$\phi_2(s) = \frac{s-\bar{s}}{2}\tilde{\kappa}(s), \quad X_2(s) = \tilde{\kappa}(s), \quad \sigma \le s \le 2\sigma \quad .$$

$$\psi_0(s) = \begin{cases} 1 & , \quad 0 \le s \le \sigma \quad , \\ 1 + 2\delta\tilde{\kappa}(s), & \sigma \le s \le 2\sigma \quad . \end{cases}$$

Then

$$\dot{\psi}_0(2\sigma-0) + \dot{\tilde{\kappa}}(2\sigma-0) \int_0^{2\sigma} \psi_0 \sin(\tilde{\theta}-2\pi)$$

$$= -2\delta - [\int_0^{\delta} \sin \tilde{\theta} + \int_{\sigma}^{2\sigma} (1 + 2\delta\tilde{\kappa})\sin \tilde{\theta}]$$

$$= -2\delta - [\delta-\delta-2\delta] = 0 \quad .$$

By Theorem 5.3 hypothesis (A) is not satisfied, thus it cannot be concluded that the hair pin configuration with the extremal interpolant $\tilde{E}$ permits perturbation. Indeed, one can show directly that if $\tilde{P}$ is replaced by the perturbed configuration $P_{\varepsilon} = \{(-\varepsilon,0), (0,\delta), (\varepsilon,0)\}$ there exists no extremal interpolant close to $\tilde{E}$ no matter how small $\varepsilon \neq 0$ is. On the other hand, if $\tilde{P}$ is replaced by ${}_{\varepsilon}P = \{(0,0), (0,\delta), (0,\varepsilon)\}$, which is also close to $\tilde{P}$, then there is an extremal interpolant ${}_{\varepsilon}E$ near $\tilde{E}$, ${}_{\varepsilon}E$ coincides with $E$ for the arc from $(0,0)$ to $(0,\delta)$, the remaining arc is the simple elastica joining $(0,\delta)$ and $(0,\varepsilon)$. Thus, we have an example of singular behavior taking place in the perturbation from $\tilde{P}$ to ${}_{\varepsilon}P$.

## §6. Special Cases of Open Extremals

We study in some detail in this section extremal P-interpolants for some special configurations.

### A. Two Point Extremal Interpolants

Let $P$ be the configuration $P = \{p_0, p_1\}$, $p_0 = 0$. For an extremal $E$ with normal representation $\bar\theta, \bar\theta(0) = 0$, we have by (3.3):

$$\text{(6.1)}\qquad \begin{aligned} &\text{(i)}\quad \bar\theta \in C^1[0,\bar s],\ \bar\theta(0) = \dot{\bar\theta}(0) = \dot{\bar\theta}(\bar s) = 0 \quad ,\\ &\text{(ii)}\quad \dot{\bar\theta}^2(s) = c^1\cos\bar\theta(s) + c^2\sin\bar\theta(s),\ 0 < s < \bar s \quad ,\\ &\text{(iii)}\quad \int_0^{\bar s}\cos\bar\theta = p_1^1,\ \int_0^{\bar s}\sin\bar\theta = p_1^2 \quad . \end{aligned}$$

(6.1i) implies $c^1 = 0$ and $c^2\sin\bar\theta(\bar s) = 0$. Clearly, either $c^2 = 0$, yielding the extremal $\bar\theta(s) \equiv 0$, $p_0 = 0$, $p_1 = (\bar s, 0)$, or, $c^2 \neq 0$. In this case we write $c^2 = -2/\ell$. Differentiation of (6.1ii) gives by Proposition 2.4 the differential equation,

$$\text{(6.2)}\qquad \ell\ddot{\bar\theta}(s) + \cos\bar\theta(s) = 0,\ 0 < s < \bar s \quad .$$

When this equation is integrated over $(0,\bar s)$ and $\dot{\bar\theta}(0) = \dot{\bar\theta}(\bar s) = 0$ is used, one obtains the first of equations (6.1iii) with $p_1^1 = 0$. From the equation,

$$(\ell/2)\dot{\bar\theta}^2 = -\sin\bar\theta \quad ,$$

it follows that $-\pi \le \bar\theta \le 0$ if $\ell > 0$ and $0 \le \bar\theta \le \pi$ if $\ell < 0$. Since the choice $\ell < 0$ amounts to a rotation through $\pi$ of the extremal corresponding to $\ell > 0$ or, to a change in orientation transforming $\bar\theta$ into $-\bar\theta$, we may assume $\ell > 0$ and $-\pi \le \bar\theta \le 0$. In this case, $p_1^2 = -d := -|p_0 - p_1|$ in the second equation of (6.1iii). We may rewrite (6.1iii):

$$\text{(6.1iii')}\qquad \int_0^{\bar s}\cos\bar\theta = 0,\ \int_0^{\bar s}\sin\bar\theta = -d \quad .$$

From (6.1ii), $\dot{\bar\theta}(\bar s) = 0$ is equivalent to $\sin\bar\theta(\bar s) = 0$. Since $\sin\bar\theta(s) \le 0$ for $0 \le s \le \bar s$ and $\bar\theta$ is continuous, there are only two possibilities: $\bar\theta(\bar s) = 0$ or $\bar\theta(\bar s) = -\pi$.

Altogether we have shown that (6.1) may be replaced by the simpler system

$$(6.3)\quad \begin{aligned} &\text{(i)}\quad \frac{\ell}{2}\,\dot{\bar\theta}^2(s) = -\sin\bar\theta(s),\quad 0 \le s \le \bar s;\quad \bar\theta(0) = 0,\quad \bar\theta(\bar s) = 0 \\ &\qquad \text{or}\quad -\pi, \\ &\text{(ii)}\quad \int_0^{\bar s} \sin\bar\theta = -d\ . \end{aligned}$$

If $s$ is interpreted as physical time, $\ell\theta$ as the displacement along a circle of radius $\ell$, then (6.3i) represents the pendulum equation with pendulum length $\ell$, unit mass and unit force downward, starting from horizontal position with velocity $0$ at $s = 0$ and reaching velocity $0$ again at $s = \bar s$ when $\bar\theta = 0$ or $-\pi$. The pendulum swings from horizontal position $\bar\theta = 0$ at $s = 0$ through one or more half-swings to horizontal position at $s = \bar s$. The kinetic analogue of the elastica equation was discovered by G. Kirchhoff; see [8, p. 399].

One interpretation of the interpolation condition (6.3ii) is that the time integral of the kinetic energy $\frac{1}{2}(\ell\dot\theta)^2$ divided by the maximum kinetic energy, $\ell$, is the prescribed "minimum time" $d$. The length of the pendulum is the main unknown of the problem.

The solution of (6.3i) for various values of $\ell$ can be derived from the solutions of the same system for $\ell = 2$. Indeed, the transformation

$$(6.4)\qquad \bar\theta(s) = \tilde\theta(\sqrt{2/\ell}\ s),\ \bar s = \sqrt{\ell/2}\ \tilde s$$

converts (5.3i) to the "normalized system":

$$(6.5)\qquad \dot{\tilde\theta}^2(s) = -\sin\tilde\theta(s),\quad 0 \le s \le \tilde s;\quad \tilde\theta(0) = 0,\quad \tilde\theta(\tilde s) = 0\ \text{ or }\ -\pi\ .$$

The solution of the pendulum equation (6.5) with $\tilde\theta(\tilde s) = -\pi$ is well known. It is explicitly given by

$$(6.6)\qquad \tilde\theta(s) = -\frac{\pi}{2} - 2\ \text{arc}\sin[2^{-1/2}\text{sn}(2^{-1/2}(s - \frac{\tilde s}{2}))],\quad 0 \le s \le \tilde s\ ,$$

where $2^{-1/2}\tilde s$ is the half-period of the Jacobi function $\text{sn}(u) = \text{sn}(u;\ 2^{-1/2})$ and arc sin is the branch of the inverse of sin with range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For $\tilde s$ and $\tilde U = U(\tilde\theta) = \int_0^{\tilde s} \dot{\tilde\theta}^2$

we find in terms of the complete elliptic integrals of the first and second kind [13]:

(6.7)(i) $$\tilde{s} = \int_0^{\pi} \frac{d\theta}{\sqrt{\sin\theta}} = 2\sqrt{2}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-t^2/2)}} = 2\sqrt{2}\,K(2^{-1/2}) \quad ,$$

and

(ii) $$\tilde{U} = \int_0^{\pi} \sqrt{\sin\theta}\, d\theta = 2\sqrt{2}\int_0^1 \sqrt{\frac{1-t^2}{1-t^2/2}} = 2\sqrt{2}[2E(2^{-1/2}) - K(2^{-1/2})] \quad .$$

The analytic continuation of $\tilde{\theta}$ (also denoted as $\tilde{\theta}$) to all of $\mathbb{R}$ is given by $\tilde{\theta}(s) = -\tilde{\theta}(-s)$ for $s < 0$ and

(6.8) $$\tilde{\theta}(s) = \begin{cases} \tilde{\theta}(2\tilde{s}-s) & \text{for } \tilde{s} \le s \le 2\tilde{s} \\ \tilde{\theta}(s-2k\tilde{s}) & \text{for } 2k\tilde{s} \le s \le 2(k+1)\tilde{s},\ k = 1,2,\ldots \end{cases} \quad .$$

It is seen that $\tilde{\theta}_{[0,2\tilde{s}]}$ also solves (6.5), the value at the new boundary point $2\tilde{s}$ being 0. In general $\tilde{\theta}_{[0,k\tilde{s}]}$ $(k = 1,2,\ldots)$ *solves (6.5), with* $\tilde{\theta}(k\tilde{s}) = -\pi$ or 0 depending on whether $k$ is odd or even. These are all the solutions of (6.5) with free right end-point.

We now use the solution $\tilde{\theta}$ of the normalized problem to express the general solution of the boundary value problem (6.3i) for fixed $\bar{s} > 0$. It is given by

(6.9)
(i) $$\bar{\theta}(s) = \tilde{\theta}(k\frac{\tilde{s}}{\bar{s}}s), \quad 0 \le s \le \bar{s} \quad ,$$

(ii) $$\ell = 2(\bar{s}/k\tilde{s})^2$$

where $k$ is any positive integer. To satisfy the remaing condition (6.3ii) we must have

$$-d = \int_0^{\bar{s}} \sin\bar{\theta}(s)ds = \int_0^{\bar{s}} \sin\tilde{\theta}(k\frac{\tilde{s}}{\bar{s}}s)ds =$$

$$\frac{\bar{s}}{k\tilde{s}}\int_0^{k\tilde{s}} \sin\tilde{\theta}(s)ds = -\frac{\bar{s}}{\tilde{s}}\tilde{U} \quad ,$$

hence

(6.10) $$\bar{s} = (\tilde{s}/\tilde{U})d \quad ,$$

independent of $k$.

We write $\bar{\theta}_k$ for the solution corresponding to $k$ ($= 1,2,\ldots$), $\ell_k$ for the corresponding parameter value in (6.3i) and $\bar{U}_k$ for the value of the curvature functional for $\bar{\theta}_k$. Thus,

$$
(6.11)\qquad
\begin{aligned}
&\text{(i)}\quad \bar{\theta}_k(s) = \tilde{\theta}\left(k\,\frac{\tilde{U}}{d}\,s\right),\quad 0 \le s \le \bar{s} = (\tilde{s}/\tilde{U})d\ ,\\
&\text{(ii)}\quad \ell_k = 2(d/k\tilde{U})^2 = 2(\bar{s}/k\tilde{s})^2,\quad k = 1,2,\ldots\ ,\\
&\text{(iii)}\quad \bar{U}_k = \int_0^{\bar{s}} \dot{\bar{\theta}}_k^2(s)\,ds = k^2\tilde{U}^2/d = k^2(\tilde{s}/\bar{s})\tilde{U}\ .
\end{aligned}
$$

All these solutions have the same arc length $\bar{s} = d\tilde{s}/\tilde{U} \approx 2.2d$. Also $\bar{\theta}_k(s) = \bar{\theta}_1(ks)$ for $0 \le s \le \bar{s}/k$ and $\bar{\theta}_k(s) = \bar{\theta}_1(ks-\bar{s})$ for $\bar{s}/k \le s \le 2\bar{s}/k$, etc. Thus the curve represented by $\bar{\theta}_k$ consists of $k$ congruent arcs, all similar (contracted by the factor $1/k$) to $\bar{\theta}_1$. The curve whose normal representation is $\bar{\theta}_k$ is characterized as that arc of the simple elastica whose endpoints are inflection points and which has $k - 1$ internal inflection points (it belongs to class $E_{(k-1)}$ in the notation of §4). For the above physical interpretation the result means: If the ratio of the time integral of the kinetic energy to the maximum kinetic energy (in a motion from the horizontal to the horizontal position) is to be the fixed number $d$ then the pendulum must have one of the lengths $\ell_k = 2(d/k\tilde{U})^2$ ($k = 1,2,\ldots$) and the pendulum makes $k$ half-swings in total time $\bar{s} = (\tilde{s}/\tilde{U})d$.

We summarize the results in

<u>Proposition 6.1</u>. There are countably many extremal P-interpolants $E_k$ interpolating a two-point configuration with $|p_0 - p_1| = d > 0$, one for each $k = 0,1,2,\ldots$ . $E_0$ is the trivial ray interpolant; $E_k$ is an arc of the simple elastica with inflection points at the terminals and $k - 1$ internal inflection points. The normal representation $\bar{\theta}_k$ of $E_k$ is given by (6.11), where $\tilde{\theta}$ is the elliptic function (6.6). Each of the $E_k$ ($k = 1,2,\ldots$) has the same length $\bar{s} = (\tilde{s}/\tilde{U})d$, where $\tilde{s}$ and $\tilde{U}$ are given by (5.7). The extremal value of the curvature functional for $E_k$ is $\bar{U}_k = k^2\bar{U}_1$ ($k = 0,1,2,\ldots$), where $\bar{U}_1 = \tilde{U}^2/d$.

<u>Remark 6.1</u>. It is shown in [5] that none of the extremals $E_1, E_2,\ldots$ provides a local minimum.

B. Ray and Rectangular Configurations.

Corollary 6.2. Suppose $P = \{0, p_1, \ldots, p_m\}$ is the "ray" configuration where $p_i = (s_i, 0)$ and $0 < s_1 < \cdots < s_m$. Countably many nontrivial extremal P-interpolants are obtained from the 2-point extremals of Proposition 6.1 as follows. Let $\bar{x}_{[0,\bar{s}_1]}$ represent any of the nontrivial extremals for the configuration $\{0, p_1\}$. Define $\bar{x}_{[\bar{s}_1, \bar{s}_2]}$ as one of the $\{p_1, p_2\}$ interpolants of Proposition 6.1 or the negative of it so that $\dot{\bar{x}}(\bar{s}_1 - 0) = \dot{\bar{x}}(\bar{s}_1 + 0)$. Continue in this way to the intervals $[\bar{s}_2, \bar{s}_3], \ldots, [\bar{s}_{m-1}, \bar{s}_m]$. The obtained curve represented by $\bar{x}$ is an extremal P-interpolant.

Corollary 6.3. Suppose $P = \{p_0, p_1, \ldots, p_m\}$ is the "rectangular" configuration where the angle between $\overline{p_{i-1}p_i}$ and $\overline{p_ip_{i+1}}$ $(i = 1, \ldots, m-1)$ is either 0 or $\pm \pi/2$. Let $\bar{x}$ represent a P-interpolant such that the segment from $p_i$ to $p_{i+1}$ $(i = 0, \ldots, m-1)$ is any of the extremals of Proposition 6.1 (including the trivial one) and so that $\dot{\bar{x}}$ is continuous. In this way countably many extremal P-interpolants *are obtained for any rectangular configuration* P.

C. Angle-Constrained Two Point Extremal Interpolants

The boundary conditions

$$\dot{\bar{x}}(0) \cdot (p_1 - p_0) = \gamma_0, \quad \dot{\bar{x}}(\bar{s}) \cdot (p_1 - p_0) = \gamma_1, \quad 0 \le \gamma_i \le |p_1 - p_0| \tag{6.12}$$

are added to the problem of Section A, replacing the zero curvature endpoint conditions.

Proposition 6.2. There are countably many extremal $\{p_0, p_1\}$-interpolants constrained by conditions (6.12) if $\gamma_0 = \gamma_1$.

Proof. Suppose $\bar{\theta}$ is the normal representation of the sought extremal. It is easily seen that one can normalize $\bar{\theta}$ so that $\bar{\theta}$ satisfies equation (6.3i), except that the values of $\bar{\theta}$ at 0 and $\bar{s}$ are not 0 and $-\pi$, but given numbers $\theta_0, \theta_1$, $0 \ge \theta_0 \ge -\frac{\pi}{2}$, $\theta_1 = \theta_0$ or $\theta_1 = \theta_0 - \pi$. It follows that the solution of the problem is a symmetric arc of the curve (6.9i), with $k$ even if $\theta_0 = \theta_1$, $k$ odd if $\theta_0 = \theta_1 + \pi$, scaled and moved so that the terminals are $p_0, p_1$.

D. Regular Configurations.

Suppose $P = \{p_0, p_1, \ldots, p_m\}$ is a "regular" configuration, by which we mean each segment $\overline{p_i p_{i+1}}$ $(i = 0, \ldots, m-1)$ is of the length $d$ and makes the same (exterior) angle $\alpha$, $\pi < \alpha \leq 2\pi$, with the following segment $\overline{p_{i+1} p_{i+2}}$. We seek an angle-constrained P-interpolant $\bar{x}$ for which

$$\dot{\bar{x}}(0) \cdot (p_1 - p_0) = \dot{\bar{x}}(\overline{s_m}) \cdot (p_m - p_{m-1}) = d \sin \frac{\alpha}{2} \quad . \tag{6.13}$$

Corollary 6.4. For each regular configuration $P = \{p_0, p_1, \ldots, p_m\}$ there are infinitely many extremal P-interpolants constrained by the condition (6.13), which consist of $m$ congruent segments.

Proof. Let $\bar{z}$ represent one of the infinitely many extremal $\{p_0, p_1\}$-interpolants $E_0$ of length $\bar{s}_1$, constrained by

$$\dot{\bar{z}}(0) \cdot (p_1 - p_0) = \dot{\bar{z}}(\bar{s}_1) \cdot (p_1 - p_0) = d \sin \frac{\alpha}{2}$$

and such that $\dot{\bar{z}}(0) \times (p_1 - p_0) = -\dot{\bar{z}}(\bar{s}_1) \times (p_1 - p_0)$. Let $\bar{x}$ represent the uniquely defined P-interpolant $E$ that extends $E_0$ by congruent pieces. Then $\bar{x}$ has continuous slope and curvature and satisfies (6.13); hence $E$ is one of the sought extremals.

E. Length-Prescribed Two Point Extremal Interpolants.

We assume now the length $\bar{s}$ of the extremal $\{p_0, p_1\}$-interpolant $\bar{E}$ is prescribed. Suppose $\bar{\theta}$ is the normal representation of $\bar{E}$. With the proper choice of the coordinate system we conclude, using Proposition 3.2, that $\bar{\theta}$ must satisfy the following conditions (assuming $\bar{s} > d$)

$$\text{(i)} \quad \dot{\bar{\theta}}(0) = \dot{\bar{\theta}}(\bar{s}) = 0 \quad ,$$

$$\text{(ii)} \quad \frac{\ell}{2} \dot{\bar{\theta}}^2(s) = -\sin \bar{\theta}(s) + \lambda, \quad 0 \leq s \leq \bar{s}, \quad \ell > 0, \quad \lambda \in \mathbb{R} \quad , \tag{6.14}$$

$$\text{(iii)} \quad \int_0^{\bar{s}} \cos \bar{\theta}(s) ds = 0, \int_0^{\bar{s}} \sin \bar{\theta}(s) ds = -d \quad .$$

Set $\bar{\theta}(0) = \theta_0 (-\frac{\pi}{2} \le \theta_0 < \frac{\pi}{2})$; then

(6.15) $$\lambda = \sin \theta_0 \quad .$$

In particular, $-1 \le \lambda < 1$. We may again interpret Equations (6.14) as describing the motion of a pendulum swinging from some position $\theta_0$ where $\dot{\theta} = 0$ in fixed time $\bar{s}$ to another position where $\dot{\theta} = 0$ and so that the time integral of the kinetic energy satisfies a certain condition.

By (6.14ii), $\dot{\bar{\theta}}(\bar{s}) = 0$ if and only if $\theta(\bar{s})$ is $\theta_0$ or $-\pi - \theta_0$. The condition $\int_0^{\bar{s}} \cos \bar{\theta} = 0$ follows from (6.14ii). Thus, (6.14) may be replaced by the simpler system

(6.16)
$$\text{(i)} \quad \frac{\ell}{2} \dot{\bar{\theta}}^2(s) = \sin \theta_0 - \sin \bar{\theta}(s), \quad 0 \le s \le \bar{s}; \quad \bar{\theta}(0) = \theta_0 \quad ,$$
$$\bar{\theta}(\bar{s}) = \theta_0 \quad \text{or} \quad -\pi - \theta_0 \quad ,$$
$$\text{(ii)} \quad \int_0^{\bar{s}} \sin \bar{\theta} = -d \quad .$$

We make the transformation (6.4) again, to obtain the normalized system:

(6.17) $$\dot{\tilde{\theta}}^2(s) = \sin \theta_0 - \sin \tilde{\theta}(s), \quad 0 \le s \le \tilde{s}, \quad \tilde{\theta}(0) = \theta_0, \quad \tilde{\theta}(\tilde{s}) = -\pi - \theta_0 \quad .$$

The solution is given by

(6.18i) $$\tilde{\theta}(s) = \tilde{\theta}(s;\theta_0) = -\frac{\pi}{2} - 2 \arcsin[q \operatorname{sn}(2^{-1/2}(s-\tilde{s}/2);\, q^2)] \quad ,$$
$$q = |\sin(\pi/4 + \theta_0/2)| \quad ,$$

where $2^{-1/2}\tilde{s}$ is the half-period of the Jacobi function $\operatorname{sn}(u;\, q)$:

(6.18ii) $$\tilde{s} = \tilde{s}(\theta_0) = -\int_{\theta_0}^{-\pi-\theta_0} \frac{d\theta}{\sqrt{\sin \theta_0 - \sin \theta}} = \int_{-\theta_0}^{\pi+\theta_0} \frac{d\theta}{\sqrt{\sin \theta_0 + \sin \theta}} =$$
$$2 \int_{-\theta_0}^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta_0 + \sin \theta}} \quad .$$

We observe that $\tilde{s}(\theta_0)$ increases monotonically from 0 to $\infty$ as $\theta_0$ varies from $-\pi/2$ to $\pi/2$. For $\tilde{d} = -\int_0^{\tilde{s}} \sin\tilde{\theta}$ and $U = \int_0^{\tilde{s}} \dot{\tilde{\theta}}^2$ we have

$$\text{(6.18iii)} \qquad \tilde{d} = \tilde{d}(\theta_0) = 2\int_{-\theta_0}^{\pi/2} \frac{\sin\theta}{\sqrt{\sin\theta_0 + \sin\theta}}\, d\theta \quad ,$$

$$\text{(6.18iv)} \qquad \tilde{U} = \tilde{U}(\theta_0) = 2\int_{-\theta_0}^{\pi/2} \sqrt{\sin\theta_0 + \sin\theta}\, d\theta \quad .$$

We also observe the identity

$$\text{(6.19)} \qquad \tilde{s}(\theta_0)\sin\theta_0 + \tilde{d}(\theta_0) = \tilde{U}(\theta_0) \quad .$$

The analytic continuation of $\tilde{\theta}$ (also denoted as $\tilde{\theta}$) to all of $\mathbb{R}$ is given, as before, by (6.8). Then we put

$$\text{(6.20)} \qquad \begin{aligned} &\text{(i)} \quad \bar{\theta}_k(s) = \tilde{\theta}(k\frac{\tilde{s}}{\bar{s}}s), \quad 0 \le s \le \bar{s}, \quad k = 1,2,\dots \ , \\ &\text{(ii)} \quad \ell_k = 2(\bar{s}/k\tilde{s})^2 \quad . \end{aligned}$$

These quantities still depend on $\theta_0$. $\theta_0$ must be determined so that the last condition

$$\text{(6.20iii)} \qquad \begin{aligned} -d &= \int_0^{\bar{s}} \sin\bar{\theta}_k(s)ds = \int_0^{\bar{s}} \sin\tilde{\theta}(k\tilde{s}s/\bar{s})ds \\ &= \frac{\bar{s}}{\tilde{s}}\int_0^{\tilde{s}} \sin\tilde{\theta}(s)ds = -\frac{\bar{s}}{\tilde{s}(\theta_0)}\tilde{d}(\theta_0) \end{aligned}$$

is satisfied. Thus $\theta_0$ is determined from the equation

$$\text{(6.21)} \qquad \frac{\tilde{d}(\theta_0)}{\tilde{s}(\theta_0)} = \frac{d}{\bar{s}} \quad ,$$

where $\tilde{d}$ and $\tilde{s}$ are given in (6.18). As $\theta_0$ varies from $-\pi/2$ to 0, $\tilde{d}(\theta_0)$ increases from 0 to $2\int_0^{\pi/2} \sin^{1/2}\theta d\theta$ and then decreases from this value to $-\infty$ as $\theta_0$ varies from

0 to $\pi/2$. Clearly $\tilde{d}(\theta_*) = 0$ for some $\theta_*$ between 0 and $\pi/2$ ($\theta_* \approx 40°$). The ratio $\tilde{d}(\theta_0)/\tilde{s}(\theta_0)$ can be seen to decrease monotonically on $[-\pi/2,\theta_*]$ with values in the entire interval $[0,1]$. Therefore, for any $0 < d < \bar{s}$, there is a unique $\theta_0$ in $[-\pi/2,\theta_*)$ such that (6.21) holds. For this unique $\theta_0$, $\tilde{s} = \tilde{s}(\theta_0)$ is determined according to (6.18ii), then $\bar{\theta}_k$ and $\ell_k$ from (6.20). Together with $\lambda = \sin\theta_0$, these quantities satisfy the original system. $\tilde{U}(\theta_0)$ is given by (6.18iv) and the value of $U$ for $\bar{\theta}_k$ by

(6.22) $$\bar{U}_k := \int_0^{\bar{s}} \dot{\bar{\theta}}_k^2(s)ds = k^2 \frac{\tilde{s}(\theta_0)}{\bar{s}} \tilde{U}(\theta_0) \quad .$$

Altogether we have proved

<u>Proposition 6.3</u>. There are countably many extremal $\{p_0,p_1\}$-interpolants $\bar{E}_1,\bar{E}_2,\ldots$ of prescribed length $\bar{s} > |p_0-p_1|$. Their normal representations are given explicitly by equations (6.18) and (6.20), with the angle of inclination $\theta_0$ at $p_0$ determined from (6.21). For the value of the functional $U$, the relation $\bar{U}_k = k^2U_1$ $(k = 1,2,\ldots)$ holds.

<u>Remark 6.2</u>. The curves of Proposition 6.2 are subarcs of inflexional elastica (cf. Remark 2.4). For illustrations see [8, p. 404, Figures 48-53].

<u>Remark 6.3</u>. In the beam interpretation the joint at $p_0$ exerts a force $R$ whose tangential component is $\sin\theta_0/\ell$ and whose normal component is $\cos\theta_0/\ell$. Thus $R$ acts along the line joining $p_0$ to $p_1$. The magnitude of the force is $1/\ell$. For fixed $\bar{s}$ and $d$, the force $R_k$ in the mode $\bar{E}_k$ has magnitude $k^2R_1$. For $\theta_0 < \pi/2$ the tangential force on the joints is a pressure, for $\theta_0 > \pi/2$ it is a pull.

§7. Examples of Closed Extremals

Let $P = \{p_0, p_1, \ldots, p_m\}$ be a configuration as in §2. If $x \in H_2^{reg}$ is such that $x(t_i) = p_i$ ( $i = 0,\ldots,m$) for some $0 < t_0 < \cdots < t_m < 1$, and besides $x^{(k)}(0+) = x^{(k)}(1-)$ for $k = 0,1,2$, we say $x$ represents an admissible closed P-interpolant with knots $p_i$. Suppose $\bar{x}$ represents an extremal closed P-interpolant, i.e. $\bar{x}$ makes the curvature functional $U$ stationary in the family of admissible closed P-interpolants. Then the graph of $\bar{x}$ is a closed curve which has continuous curvature everywhere, and the curvature is continuously differentiable at all points other than the knots.

In this paragraph we examine four classes of closed extremals:

A. Closed extremals of prescribed length with no knots.

B. Closed symmetric extremals with two knots.

C. Closed extremals for rectangular configurations.

D. Closed extremals for regular polygons.

A. If $\bar{x}$ represents a closed extremal with no knots, of prescribed length $\bar{s} > 0$, parametrized with respect to arc length, $\kappa(s)$ its curvature at $s$, then one finds, as in Proposition 2.3, that

$$(7.1)\qquad \begin{aligned} &\text{(i)} \quad \bar{x} \in C^{\infty}[0,s], \quad x^{(k)}(0) = x^{(k)}(\bar{s}), \quad k = 0,1,2, \\ &\text{(ii)} \quad 2\dddot{\bar{x}} + 3\kappa^2\dot{\bar{x}} - \lambda\dot{\bar{x}} = c, \; c \in \mathbb{R}^2 \quad , \\ &\text{(iii)} \quad \lambda = \frac{1}{\bar{s}}\int_0^{\bar{s}} \kappa^2 \quad . \end{aligned}$$

Conversely if $\bar{x}$ satisfies (7.1) then $\bar{x}$ represents a closed extremal with no knots, of prescribed length $\bar{s}$, parametrized by arc length. For the normal representation $\bar{\theta}$, where $\bar{x}^1(s) = \int_0^s \cos\bar{\theta}$, $\bar{x}^2(s) = \int_0^s \sin\bar{\theta}$, (7.1) gives

$$
(7.2)\quad
\begin{aligned}
&\text{(i)}\quad \bar\theta \in C^\infty[0,\bar s],\ \int_0^{\bar s}\cos\bar\theta = \int_0^{\bar s}\sin\bar\theta = 0 ,\\
&\qquad \bar\theta(\bar s) = \bar\theta(0) + 2k\pi\ (k = 0,\pm1,\ldots);\ \dot{\bar\theta}(0) = \dot{\bar\theta}(\bar s) ,\\
&\text{(ii)}\quad \dot{\bar\theta}^2 = c^1\cos\bar\theta + c^2\sin\bar\theta + \lambda ,\\
&\text{(iii)}\quad \lambda = \frac{1}{\bar s}\int_0^{\bar s}\dot{\bar\theta}^2 .
\end{aligned}
$$

<u>Proposition 7.1</u>. For each $k = 1,2,\ldots$ there exist exactly two closed extremals with no knots of prescribed length $\bar s > 0$. These are the circle of radius $\bar s/(2k\pi)$ transversed $k$ times and a contracted figure eight configuration traversed $k$ times.

<u>Proof</u>. We can omit (iii) in (7.2) since it follows from (i) and (ii). We write $A\sin(\bar\theta + \alpha)$ with $A \geq 0,\ \alpha \in T$ for $c^1\cos\bar\theta + c^2\sin\bar\theta$. If $\bar\theta$ represents an extremal then $\bar\theta - \alpha$ represents the same extremal rotated by the angle $\alpha$. Therefore, (7.2ii) may be replaced by $\dot{\bar\theta}^2 = A\sin\bar\theta + \omega_0^2$, where $\omega_0^2 > 0$. ($\omega_0^2 = 0$ means $\lambda = 0$, which is impossible by (7.2iii)). We may also assume $k = 0,1,\ldots,$ in (7.2i). Thus, (7.2) is replaced by

$$
(7.3)\quad
\begin{aligned}
&\text{(i)}\quad \bar\theta \in C^\infty[0,\bar s],\ \int_0^{\bar s}\cos\bar\theta = \int_0^{\bar s}\sin\bar\theta = 0 ,\\
&\qquad \bar\theta(\bar s) = \bar\theta(0) + 2k\pi\ (k = 0,1,2,\ldots);\ \dot\theta(0) = \dot{\bar\theta}(\bar s) ,\\
&\text{(ii)}\quad \dot{\bar\theta}^2 = A\sin\bar\theta + \omega_0^2,\ A \geq 0,\ \omega_0 > 0 .
\end{aligned}
$$

Case 1. $A = 0$. In this case we may assume $\theta_0 = 0$. Then $\bar\theta(s) = \omega_0 s$ and $\omega_0\bar s = 2k\pi$, hence $\omega_0 = 2k\pi/\bar s$. For $k = 1,2,\ldots,\bar\theta_k(s) = 2k\pi s/\bar s$ satisfies all conditions. $\bar\theta_k$ represents a circle of radius $\bar s/2k\pi$, traversed $k$ times. The value of $U$ for $\bar\theta_k$ is $(2k\pi/\bar s)^2$.

Case 2. $A > 0,\ \omega_0^2 > A$. We may assume $A = 1$ since if $\bar\theta$ is a solution of (7.3) for $A > 0$, of length $\bar s > 0$, then $\tilde\theta$ defined by $\bar\theta(A^{-1/2}s)$ is a solution for $A = 1,\ \omega_0^2 = A^{-1}\omega_0^2 > 1$, of length $\tilde s = A^{1/2}\bar s$. If $\bar\theta$ satisfies (7.3) then $\bar\theta(s)$ is uniquely defined by

$$
(7.4)\qquad s = \int_{\theta_0}^{\bar\theta(s)} \frac{d\varphi}{\sqrt{\omega_0^2 + \sin\varphi}} ,
$$

$k$ in (7.3i) must be positive, and $\omega_0^2$ is uniquely defined by

$$\bar{s} = \int_0^{2k\pi} \frac{d\varphi}{\sqrt{\omega_0^2 + \sin\varphi}} \quad .$$

For $\bar{\theta}$ defined in this way we have, after a change of variable,

$$\int_0^{\bar{s}} \sin\bar{\theta}(s)ds = \int_{-k\pi}^{k\pi} \frac{\sin\varphi}{\sqrt{\omega_0^2 + \sin\varphi}} d\varphi$$

$$= k[\int_0^{\pi} \frac{\sin\varphi}{\sqrt{\omega_0^2 + \sin\varphi}} d\varphi - \int_0^{\pi} \frac{\sin\varphi}{\sqrt{\omega_0^2 - \sin\varphi}} d\varphi] < 0 \quad ,$$

which contradicts (7.3i). Thus no solution exists for $\omega_0^2 > A$.

Case 3. $A > 0$, $\omega_0^2 < A$. We may again assume $A = 1$, thus $\omega_0 < 1$. Conveniently replace $\theta$ by $\theta + \pi$, and write $\sin\theta_*$ for $\omega_0^2$, with $0 < \theta_* < \frac{\pi}{2}$. Thus (7.3ii) is replaced by

(7.3ii') $$\dot{\bar{\theta}}^2 = \sin\theta_* - \sin\bar{\theta} \quad .$$

In this case $-\pi - \theta_* \le \bar{\theta}(s) \le \theta_*$, thus we must have $k = 0$ in (7.3i). Since $\bar{\theta}(s)$ cannot be monotone, we must have $\dot{\bar{\theta}}(s) = 0$ and $\sin\bar{\theta}(s) = \sin\theta_*$ for some $s$; it is no restriction to assume that this happens for $s = 0$, and $\bar{\theta}(0) = \theta_*$, $\dot{\bar{\theta}}(0) = 0$. As $s$ increases from 0 to some $s_*$, $\bar{\theta}(s)$ decreases from $\theta_*$ to $-\pi - \theta_*$, when $\dot{\bar{\theta}}(s_*) = 0$. Since $-\pi - \theta_* \neq \theta_*$ the curve cannot be closed yet, and as $s$ increases further, $\bar{\theta}(s)$ increases up to the value $\theta_*$, which is attained for some $s = s_{**}$, and $\dot{\bar{\theta}}(s_{**}) = 0$. If we set $\theta_1(t) = \bar{\theta}(2s_*-t)$, we see that $\theta_1$ satisfies (7.3ii') and $\theta_1(s_*) = \bar{\theta}(s_*)$, hence $\theta_1 = \bar{\theta}$, i.e.

$$\bar{\theta}(s_* + t) = \bar{\theta}(s_* - t)$$

and, in particular, $s_{**} = 2s_*$. Thus the curve obtained is symmetric w.r.t. the point $s = 0$. If we put $\theta_2(s) = -\pi - \bar{\theta}(s_*-s)$ then we see that $\theta_2$ satisfies (7.3ii') and $\theta_2(0) = \bar{\theta}(0) = \theta_*$, hence

$$\bar{\theta}(s_* - t) = -\pi - \bar{\theta}(t) \quad ,$$

the curve obtained is symmetric with respect to the line $\theta = 0$ and $\bar{\theta}(s_*/2) = -\pi/2$.

The curve will be closed iff the conditions $\int_0^{s_{**}} \cos\bar{\theta} = 0$, $\int_0^{s_{**}} \sin\bar{\theta} = 0$ are satisfied. The first equation follows directly from the symmetry of the curve. We are left with

$$0 = \int_0^{s_{**}} \sin\bar{\theta}(s)ds = 4\int_{-\pi/2}^{\theta_*} \frac{\sin\varphi}{\sqrt{\sin\theta_* - \sin\varphi}} d\varphi = 0 \quad . \tag{7.3i'}$$

Let the last integral be denoted as $I(\theta_*)$. Clearly $I(0) = -\frac{1}{2}\tilde{U} < 0$ and $I(\frac{\pi}{2}) = +\infty$. Thus, there is a value $\theta_*$ between $0$ and $\pi/2$ for which (7.3i') holds, and it is easily seen that there is only one such value (approximately $\theta_* = 40°$). With this value of $\theta_*$ we have obtained a closed extremal $E^*$ of length $s_{**}$. It is an analytic curve, crossing itself at $s = \frac{1}{2}s_{**}$, and consists of 2 congruent loops, each symmetric w.r.t. the same axis (an illustration appears in [8, p. 404] as an example of an inflexional elastica). By proper scaling the curve will have the prescribed length $\bar{s}$. The differential equation for the normal representation of the curve $E_1^*$ is

$$a^2\dot{\theta}_1^{*2} = \sin\theta_* - \sin\theta_1^*,\ \theta_1^*(0) = \theta_*,\ \theta_1^*(\frac{\bar{s}}{4}) = \frac{-\pi}{2} \quad .$$

Thus, the inverse function $\theta \to s(\theta)$ is given by

$$s(\theta) = a\int_\theta^{\theta_*} \frac{d\varphi}{\sqrt{\sin\theta_* - \sin\varphi}},\quad \frac{-\pi}{2} \le \theta \le \theta_*$$

where the constant $a$ is determined from

$$\frac{\bar{s}}{4} = a\int_{-\pi/2}^{\theta_*} \frac{d\varphi}{\sqrt{\sin\theta_* - \sin\varphi}} \quad ,$$

The other extremals $E_2^*, E_3^*, \ldots$ in this sequence are obtained by traversing $E_1^*$ $2, 3, \ldots$ times with scale factor $\frac{1}{2}, \frac{1}{3}, \ldots$ , thus their normal representations satisfy

$$\theta_k^*(s) = \theta_1^*(ks),\quad 0 \le s \le \bar{s} \quad .$$

Case 4. $A_0 > 0$, $\omega_0^2 = A$. In this case the solution of (7.3i) is monotonically increasing in $s$ but does not attain $\theta_0 + 2\pi$ for finite $s$.

<u>Remark 7.1</u>. The restriction of $\theta_1^*(s)$ to $[0,\frac{\bar{s}}{2}]$ represents a length-prescribed extremal (length = $\bar{s}/2$) interpolating the "<u>loop configuration</u>" $\{p_0,p_1\}$ with $p_0 = p_1$. This is not a closed extremal, although it is a closed curve. Each $\theta_1^*(ks)$ $(k = 1,2,\ldots;\ 0 \leq s \leq \frac{\bar{s}}{2})$ can also be considered as the n.r. of such an interpolating extremal. The curvature functional (potential energy) for this extremal is seen to have the value

$$\frac{4k^2}{\bar{s}}\left[\int_{-\pi/2}^{\theta_*} \frac{d\varphi}{\sqrt{\sin\theta_* - \sin\varphi}}\right]^2 \sin\theta_* \quad .$$

Clearly this function of $\bar{s}$ has no stationary value. It follows that there exists no extremal interpolant (unconstrained) for the loop configuration.

<u>Remark 7.2</u>. The length-prescribed extremal interpolating the loop configuration can also be obtained as the limiting case of the extremal of Sec. 6.E as $d \to 0$. There it was pointed out that $\tilde{d}(\theta_*) = 0$ for $\theta_*$ satisfying (see 6.18iii)

$$0 = \int_{-\theta_*}^{\pi/2} \frac{\sin\theta}{\sqrt{\sin\theta_* + \sin\theta}}\, d\theta \quad .$$

Clearly, this is the above condition (7.3i').

B. We turn to the problem of closed extremals $\mathring{E}$ with two knots. We consider only extremals that are symmetric with respect to the line joining the two given knots. We assume $p_0 = (0,0)$, $p_1 = (0,-d)$ with $d > 0$ are the knots and that $\mathring{x} = (\mathring{x}^1,\mathring{x}^2)$ represents the extremal $\mathring{E}$, parametrized by arc length. If $\mathring{s}$ is the length of $\mathring{E}$ and $\mathring{x}(0) = p_0$, then $\mathring{x}^{(k)}(\bar{s}) = \mathring{x}^{(k)}(0)$ for $k = 0,1,2,$ and because of the symmetry, $\mathring{x}(\bar{s}/2) = p_1$. Thus, we may assume

$$\begin{aligned}
&\mathring{x}^1(s) = -\mathring{x}^1(\mathring{s}-s), \quad \mathring{x}^2(s) = \mathring{x}^2(\mathring{s}-s), \quad 0 \leq s \leq \mathring{s} \quad ,\\
(7.5)\ (7.5)\qquad &\mathring{x}^1(0) = \mathring{x}^1(\tfrac{\mathring{s}}{2}) = \mathring{x}^1(\mathring{s}) = 0; \quad \mathring{x}^2(0) = \mathring{x}^2(\mathring{s}) = 0, \quad \mathring{x}^2(\tfrac{\mathring{s}}{2}) = -d \quad ,\\
&\dot{\mathring{x}}^1(0) = \dot{\mathring{x}}^1(\mathring{s}) = 1, \quad \dot{\mathring{x}}^2(\tfrac{\mathring{s}}{2}) = \pm 1; \quad \dot{\mathring{x}}^2(0) = \dot{\mathring{x}}^2(\tfrac{\mathring{s}}{2}) = \dot{\mathring{x}}^2(\mathring{s}) = 0 \quad .
\end{aligned}$$

If $\mathring{c}$ is the normal representation of $\mathring{E}$ then by (7.5) and Proposition 3.1,

$$\text{(i)} \quad \cos\mathring{\theta}(s) = \cos\mathring{\theta}(\mathring{s}-s), \quad \sin\mathring{\theta}(s) = -\sin\mathring{\theta}(\mathring{s}-s) \quad , \quad 0 \le s \le \mathring{s} \quad ,$$

$$\text{(ii)} \quad \int_0^{\mathring{s}/2} \cos\mathring{\theta} = 0, \quad \int_0^{\mathring{s}/2} \sin\mathring{\theta} = -d, \quad \int_0^{\mathring{s}} \cos\mathring{\theta} = 0 \quad , \quad \int_0^{\mathring{s}} \sin\mathring{\theta} = 0 \quad ,$$

(7.6)
$$\text{(iii)} \quad \mathring{\theta}(0) = 0, \quad \mathring{\theta}(\tfrac{\mathring{s}}{2}) = j\pi, \quad \mathring{\theta}(\mathring{s}) = 2j\pi; \quad j = 0 \text{ or } -1 \quad .$$

$$\text{(iv)} \quad \dot{\mathring{\theta}}(0) = \dot{\mathring{\theta}}(\tfrac{\mathring{s}}{2}) = \dot{\mathring{\theta}}(\mathring{s}) = 0 \quad ,$$

$$\text{(v)} \quad \dot{\mathring{\theta}}^2(s) = \lambda_1^1\cos\mathring{\theta}(s) + \lambda_1^2\sin\mathring{\theta}(s), \quad 0 \le s \le \mathring{s}/2 \quad ,$$
$$= \lambda_2^1\cos\mathring{\theta}(s) + \lambda_2^2\sin\mathring{\theta}(s), \quad \mathring{s}/2 \le s \le \mathring{s} \quad .$$

$s = \mathring{s}/2$ in (v) gives $\lambda_1^1 = \lambda_2^1 = 0$; substitution of (i) in (v) gives $\lambda_1^2 = -\lambda_2^2 = -\lambda$. Thus (7.6v) becomes

(7.6v')
$$\dot{\mathring{\theta}}^2(s) = -\lambda \sin\mathring{\theta}(s), \quad 0 < s \le \mathring{s}/2 \quad ,$$
$$= +\lambda \sin\mathring{\theta}(s), \quad \mathring{s}/2 \le s \le \mathring{s} \quad .$$

When the conditions $\mathring{\theta}(0) = 0$, $\mathring{\theta}(\mathring{s}/2) = 0$ or $-\pi$, $\dot{\mathring{\theta}}(0) = \dot{\mathring{\theta}}(\mathring{s}/2) = 0$ are taken together with (7.6v'), it is seen that $\mathring{\theta}_{[0,\bar{s}/2]}$ is one of the functions $\theta_k$ of §6, with $\bar{s}$ replaced by $\mathring{s}/2$. The symmetry condition gives for $\mathring{\theta}_{[\mathring{s}/2,\mathring{s}]}$:

$$\mathring{\theta}(s) = -2j\pi - \mathring{\theta}(\mathring{s}-s), \quad \mathring{s}/2 \le s \le \mathring{s} \quad .$$

Thus we have found all solutions of system (7.6) and have proved

Proposition 7.2. There are countably many closed extremals $\mathring{E}_1, \mathring{E}_2, \ldots$ with n.r. $\mathring{\theta}_1, \mathring{\theta}_2, \ldots$ with two knots which are symmetric with respect to the line joining the knots. $\mathring{E}_k$ is obtained from the open extremal $E_k$ of Proposition 6.1 by reflection at the line joining the knots. Each $\mathring{E}_k$ has the same length $2\bar{s}$, where $\bar{s}$ is the length of the open $E_k$. The value of the curvature functional $U$ for the extremal $\overset{\circ}{E}_k$ is $2k^2\bar{U}_1$ where $\bar{U}_1$ is the value for the open $E_1$.

C. In this section we consider rectangular configurations as defined in Corollary 6.2.

Proposition 7.3. Let $P = \{p_0,\ldots,p_m,p_0\}$ be a rectangular configuration as in Corollary 6.2. There exist countably many closed extremals with knots at $p_0,\ldots,p_m$.

Proof. Since $P$ is closed there must be an even number of right angles between consecutive segments $\overline{p_{i-1}p_i}$, $\overline{p_ip_{i+1}}$. To connect $p_i$ to $p_{i+1}$ we use either the trivial extremal or one of the 2-point open extremals of Proposition 6.1, with the proviso that we switch from one class of extremals to the other if the angle at $p_i$ is a right angle, otherwise (if the angle is 0) no switch is made. It is easy to see that infinitely many closed P-interpolants with continuous curvature everywhere can be obtained in this way.

D. Let $p_1,\ldots,p_m$ be the vertices of a regular polygon ordered as they come when the polygon is traversed counter-clockwise. Define $p_i$ for $i > m$ by periodicity: $p_i = p_{i-m}$. Let $P_{m,k} = \{p_1,p_{1+k},\ldots,p_{1+mk}\}$ for $k = 1,2,\ldots$ . $P_{m,k}$ is a configuration of the kind that Corollary 6.4 applies to, and if the construction used there is applied to $P_{m,k}$ one obtains closed extremals. Thus we have

Proposition 7.4. For each regular configuration $P_{m,k}$ as described above there are infinitely many closed extremal P-interpolants, each composed of congruent segments. For each $k$, there is precisely one such extremal whose intersection with the polygonal path connecting the points of $P_{m,k}$ is precisely $P_{m,k}$. For $k = 1$, this extremal $E^*_m$ circumscribes the polygon counter-clockwise and its representation $x^*_m$ satisfies

$$(p_{i+1} - p_i) \cdot \dot{x}^*_m(s_i) = |p_{i+1} - p_i|\cos\frac{\pi}{m}, \quad i = 1,\ldots,m \ .$$

If the polygon is inscribed in a unit circle, each of the $m$ arcs of $E^*_m$ may be expressed in terms of the inverse of its normal representation:

(7.7i) $$s_m^*(\theta) = \frac{2\sin\frac{\pi}{m} F(\frac{1}{2}\sqrt{2},\psi)}{2E(\frac{1}{2}\sqrt{2},\beta_m) - F(\frac{1}{2}\sqrt{2},\beta_m)}, \quad \frac{\pi}{2} - \frac{\pi}{m} \le \theta \le \frac{\pi}{2} + \frac{\pi}{m} \quad ,$$

where $\cos\psi = \sqrt{\sin\theta}$ and $\cos\beta_m = \sqrt{\cos\pi/m}$. In this case, the length $s_m^*$ and the energy $U_m^*$ of $E_m^*$ are given explicitly by,

(7.7ii) $$s_m^* = \frac{2m\sin\frac{\pi}{m} F(\frac{1}{2}\sqrt{2},\beta_m)}{2E(\frac{1}{2}\sqrt{2},\beta_m) - F(\frac{1}{2}\sqrt{2},\beta_m)} \quad ,$$

and

(7.7iii) $$U_m^* = \frac{4m}{\sin\frac{\pi}{m}}[2E(\frac{1}{2}\sqrt{2},\beta_m) - F(\frac{1}{2}\sqrt{2},\beta_m)]^2 \quad .$$

Finally,

(7.8) $$\frac{s_m^*(\theta)}{\theta} \to 1 \quad \text{as} \quad m \to \infty \quad ,$$

so that the extremals have the unit circle as limit.

<u>Proof</u>. We sketch the verification of (7.7). Starting from the differential equation

$$\gamma_m^2\dot\theta^2 = \sin\theta \quad ,$$

we obtain

$$s_m^*(\theta) = \gamma_m \int_{\alpha_m}^{\theta} \frac{d\varphi}{\sqrt{\sin\varphi}}, \quad \alpha_m \le \theta \le \pi-\alpha_m, \quad \alpha_m = \frac{\pi}{2} - \frac{\pi}{m} \quad .$$

Since the distance between adjacent points of $P_{m,1}$ is $2\cos\alpha_m$, the constant $\gamma_m$ is determined from

$$\gamma_m = \cos\alpha_m / \int_{\alpha_m}^{\pi/2} \sqrt{\sin\varphi}\, d\varphi$$

$$= \frac{\sin\frac{\pi}{m}}{\sqrt{2}[2E(\frac{1}{2}\sqrt{2},\beta_m) - F(\frac{1}{2}\sqrt{2},\beta_m)]} \quad .$$

This gives (7.7i). (7,7ii) is immediate and (7.7iii) follows from,

$$U_m^* = 2m \int_0^{s_m^*/2m} \dot{\theta}^2 ds = \frac{2m}{\gamma_m} \int_m^{\pi/2} \sqrt{\sin \varphi}\, d\varphi \ .$$

The calculation (7.8) is routine.

Remark 7.3. It is shown in [5] that the extremals $E_m^*$ are stable, i.e. they provide a local minimum for the curvature functional.

## Appendix

Let $P = \{p_0,\ldots,p_m\}$, be given and let $L_0 = \sum_{i=1} |p_{i+1}-p_i|$. We assume that $P$ is not collinear.

<u>Theorem</u>. For every $L > L_0$ there exists a length-prescribed extremal P-interpolant of length $L$ satisfying Definition 2.2.

<u>Proof</u>. The existence of a function $\bar{x}$, parametrized with respect to arc length, for which

$$U(\bar{x}) = \min\{U(x) : x \text{ is an admissible P-interpolant of length } L\}$$

is demonstrated in [6]. (The modification required for the ordering of the points in $P$ is trivial.) If $X$ denotes the closed subspace of $H_2[0,L]$ consisting of those functions vanishing at the knots,

$$0 \le \bar{s}_0 < \bar{s}_1 < \cdots < \bar{s}_m \le L \quad ,$$

of $\bar{x}$, i.e., at those points $\bar{s}_i$ for which $\bar{x}(\bar{s}_i) = p_i$, $i = 0,\ldots,m$, define $f$ to be the mapping of $X$ into $\mathbb{R}$ obtained by $f(y) = U(\bar{x} + y)$ and let $H$ be the function such that $H(y) = S(\bar{x} + y) - L$, where $S$ is the usual length functional (cf. (2.13ii)). Clearly,

$$f(0) = \min\{f(y) : H(y) = 0\} \quad ,$$

and $H'(0)$ is surjective, since $P$ is assumed non-collinear. The result follows from the Lagrange multiplier rule (see, e.g., [9, Theorem 1, p. 243]).

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MG/JJ/jvs

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# REPORT DOCUMENTATION PAGE

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| | |
|---|---|
| 1. REPORT NUMBER: #2024 | 2. GOVT ACCESSION NO.: AD-A083 867 |
| 3. RECIPIENT'S CATALOG NUMBER | |
| 4. TITLE (and Subtitle): EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINEAR INTERPOLATING SPLINE CURVES | 5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s): Michael Golomb and Joseph Jerome | 8. CONTRACT OR GRANT NUMBER(s): DAAG29-75-C-0024; MPS 74-02292 A01 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 6 - Spline Functions and Approximation Theory |
| 11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below | 12. REPORT DATE: November 1979 |
| | 13. NUMBER OF PAGES: 64 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | 15. SECURITY CLASS. (of this report): UNCLASSIFIED |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office, P. O. Box 12211, Research Triangle Park, North Carolina 27709
and
National Science Foundation, Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear spline curves, elastica, manifolds of extremals, perturbation stability, mode, ray configuration, rectangular configuration

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)


A detailed global and local analysis of smooth solutions of the variational problem

(1i) $$\delta \int_0^{\bar{s}} \kappa^2(s)\,ds = 0 ,$$

subject to position function constraints

(1ii) $$x(s_i) = p_i, \quad 0 \le s_0 < s_1 < \cdots < s_m \le \bar{s} ,$$


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ABSTRACT (continued)

is carried out. Here $\{p_i\}_0^m \subset \mathbb{R}^2$ is prescribed, $x$ is a vector-valued function with curvature $\kappa(s)$ at arc length $s$ and the interpolation nodes $s_i$ are free. Problem (1) may be viewed as the mathematical formulation of the draftsman's technique of curve fitting by mechanical splines.

Although most of the basic equations satisfied by these nonlinear splines curves have been known for a very long time, calculation via elliptic integral functions has been hampered by a lack of understanding concerning what precise information must be specified for the stable determination of a smooth, unique interpolant modelling the thin elastic beam. In this report, sharp characterizations are derived for the extremal interpolants as well as structure theorems in terms of inflection point modes which guarantee uniqueness and well-posedness.

A certain type of stability is introduced and studied and shown to be related to (linearization) concepts associated with piecewise cubic spline functions, which have been studied for decades as a simplication of the nonlinear spline curves. Many examples are introduced and studied.